AIAA JOURNAL
Vol. 42, No. 4, April 2004

Unified Theory for the Dynamics and Control of Maneuvering Flexible Aircraft

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This work represents a new paradigm for the dynamics and control of maneuvering flexible aircraft. Using the system concept, the theory integrates seamlessly all the necessary material from the areas of analytical dynamics, structural dynamics, aerodynamics, and controls. It includes automatically all six rigid-body degrees of freedom and elastic deformations, as well as the gravity, propulsion, aerodynamic, and control forces, in addition to forces of an external nature, such as gusts. The seamless integration is achieved by using the same reference frame and the same variables to describe the aircraft motions and the forces acting on it, including the aerodynamic forces. The formulation is modular in nature, in the the sense that the structural model, the aerodynamic theory, and the controls method can be replaced by any other ones to better suit different types of aircraft, provided certain criteria are satisfied. A perturbation approach permits the separation of the equations of motion into a flight dynamics problem for the maneuvering aircraft rigid-body translations and rotations and an extended perturbation problem for the elastic deformations and perturbations in the rigid-body variables, where the second problem is subject to inputs from the maneuvering aircraft. The formulation is ideally suited for unmanned aerial vehicles (UAVs), and in particular for autonomous UAVs requiring autopilots. A numerical example presents a variety of time simulations of rigid-body perturbations and elastic deformations for two cases, 1) a steady level flight and 2) a level steady turn maneuver. All the time simulations were carried out on a 1-GHz personal computer, which is particularly important for autonomous UAVs, as the required onboard computer is likely to be much closer to a personal computer than to a multiprocessor supercomputer. The unified theory is expected to stimulate new interest in aeronautics research, ultimately providing a useful tool for the analysis and design of flexible aircraft.

Nomenclature

A, B = coefficient matrices in the state equations
C = matrix relating the output vector to the state vector
C_e = matrix of direction cosines between $x_e y_e z_e$ and $x_f y_f z_f$
C_f = matrix of direction cosines between $x_f y_f z_f$ and XYZ
$C_{ui}, C_{\psi i}$ = damping matrices for the bending and torsion of component i
C_w = matrix of direction cosines between $x_w y_w z_w$ and $x_f y_f z_f$
$c_{ui}, c_{\psi i}$ = bending and torsion damping functions for component i
d_i = drag force per unit span of component i
E_f = matrix relating Eulerian velocities to angular quasi-velocities
EI_i, GJ_i = flexural and torsional rigidities for component i
$\boldsymbol{e}$ = observer error vector
$\boldsymbol{F}, \boldsymbol{M}$ = resultant of gravity, aerodynamic, propulsion, and control force and moment vectors
$\boldsymbol{F}_E$ = engine thrust vector
$\boldsymbol{F}_{\text{ext}}$ = external disturbing force vector
$\hat{\mathcal{F}}_{ui}, \hat{\mathcal{F}}_{\psi i}$ = Rayleigh's dissipation function densities for body i
$\boldsymbol{f}_{ai}, \boldsymbol{f}_{si}$ = distributed aerodynamic force vectors for component i
$\boldsymbol{f}_{gi}$ = distributed gravity force vector for component i
$\boldsymbol{f}_i$ = distributed force vector for component i due to gravity, aerodynamics, and control
G = control gain matrix
g = gravitational constant
I = identity matrix
J = inertia matrix for the deformed aircraft
K_o = observer gain matrix
$K_{ui}, K_{\psi i}$ = stiffness matrices for the bending and torsion of component i
L = Lagrangian for the whole aircraft
$\hat{L}_i$ = Lagrangian density for body i exclusive of strain energy
$\mathcal{L}_{ui}, \mathcal{L}_{\psi i}$ = matrices of stiffness differential operators for body i
ℓ_i = lift force per unit span of component i
M = discrete system mass matrix
M_{ij} = mass submatrices
m = total aircraft mass
m_i = mass of component i
O_i = origin of body axes for component i
$\boldsymbol{p}$ = momentum vector for the whole aircraft
$\boldsymbol{p}_{ui}, \boldsymbol{p}_{\psi i}$ = momentum vectors for the bending and torsion of component i
$\boldsymbol{p}_{Vf}, \boldsymbol{p}_{\omega f}$ = momentum vectors for aircraft rigid-body translation and rotation
$\boldsymbol{Q}_{ui}, \boldsymbol{Q}_{\psi i}$ = vectors of generalized forces for the bending and torsion of component i
$\boldsymbol{q}_{ui}, \boldsymbol{q}_{\psi i}$ = vectors of generalized coordinates for the bending and torsion of component i
$\boldsymbol{R}_f$ = position vector of origin O_f of $x_f y_f z_f$ relative to XYZ, $\boldsymbol{R}_f(X_f, Y_f, Z_f)$
$\boldsymbol{r}_{fw}$ = radius vector from O_f to O_w
$\boldsymbol{r}_i$ = nominal position vector of a point on component i
$\tilde{S}$ = matrix of first moments of inertia of the deformed aircraft

Presented as Papers 2002-4747 and 2002-5055 at the AIAA Guidance, Navigation, and Control Conference, Monterey, CA, 5–8 August 2002; received 27 March 2003; revision received 8 December 2003; accepted for publication 19 December 2003.

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s_i = side force per unit span of component i
$\boldsymbol{s}_{ui}, \boldsymbol{s}_{\psi i}$ = vectors of generalized velocities for the bending and torsion of component i
T = total kinetic energy
T_i = kinetic energy of component i
$\hat{\boldsymbol{U}}_i, \hat{\boldsymbol{\Psi}}_i$ = resultant of gravity, aerodynamic, propulsion, and control force and moment density vectors for body i
$\boldsymbol{u}_i, \boldsymbol{v}_i$ = elastic displacement and velocity vectors for body i
V = total potential energy
$\boldsymbol{V}$ = discrete system velocity vector
$\boldsymbol{V}_f, \boldsymbol{\omega}_f$ = translational and angular quasi-velocity vectors of $x_f y_f z_f$
$\tilde{V}_f, \tilde{\omega}_f$ = skew symmetric matrices derived from $\boldsymbol{V}_f, \boldsymbol{\omega}_f$ (see Ref. 12)
$\bar{\boldsymbol{V}}_i(\boldsymbol{r}_i, t)$ = velocity vector of a point on component i
XYZ = inertial axes
$x_i y_i z_i$ = body axes of component i
$\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}$ = zero- and first-order state vectors
$\hat{\boldsymbol{x}}$ = observer state vector
$\boldsymbol{y}$ = measurement vector
$\boldsymbol{y}^{(1)}$ = output vector
$\boldsymbol{\alpha}_{fw}$ = angular velocity of the fuselage at $\boldsymbol{r}_{fw}$ due to torsion
α_i = angle of attack of the lift force for component i
β_i = angle of attack of the side force for component i
$\delta_a^{(0)}, \delta_a^{(1)}$ = zero- and first-order aileron angles
$\delta_e^{(0)}, \delta_e^{(1)}$ = zero- and first-order elevator angles
$\delta\boldsymbol{q}_{ui}, \delta\boldsymbol{q}_{\psi i}$ = vectors of virtual generalized displacements for component i
$\delta\boldsymbol{R}_f^*, \delta\boldsymbol{\theta}_f^*$ = vectors of virtual generalized displacements in rigid-body translation and rotation
$\delta\bar{\boldsymbol{R}}_i^*$ = vector of virtual quasi-displacements
$\delta(\boldsymbol{r}-\boldsymbol{r}_E)$ = spatial Dirac delta function at $\boldsymbol{r}=\boldsymbol{r}_E$
$\delta_r^{(0)}, \delta_r^{(1)}$ = zero- and first-order rudder angles
$\delta\bar{W}$ = virtual work for the whole aircraft
$\boldsymbol{\theta}_f$ = symbolic vector of Eulerian angles between $x_f y_f z_f$ and XYZ
ρ_i = mass density of component i
$\Phi_{ui}, \Phi_{\psi i}$ = matrices of shape functions for the bending and torsion of component i
ψ, θ, ϕ = Eulerian angles from inertial axes XYZ to body axes $x_f y_f z_f$
$\boldsymbol{\psi}_i, \boldsymbol{\alpha}_i$ = elastic angular displacement and velocity vectors for body i
$\boldsymbol{\Omega}_{fw}$ = angular velocity vector of the fuselage at $\boldsymbol{r}_{fw}$ due to bending

I. Introduction

THE motion of flexible aircraft can be described by one of the two types of reference frames:

A. Fixed in the Undeformed Body

In this case, it is convenient to define the translation of the origin of the reference frame and the rotation of the reference frame as the rigid-body translation and rotation of the aircraft, and regard any displacement relative to the reference frame as elastic deformation.

B. Moving Relative to the Undeformed Body

In this case, it is common to choose the reference axes so that the linear momentum and angular momentum vectors due to elastic deformations vanish; axes satisfying these constraints are called *mean axes*. Because the elastic deformations depend in general on time, the mean axes are continuously moving relative to the undeformed aircraft. It appears that the concept of mean axes originated in the 1960s in connection with flexible spacecraft dynamics, when mean axes were known to as a "floating reference frame." Of particular interest are mean axes with the origin at the system mass center because in this case the three types of motion, namely, the translations of the reference frame, the rotations of the reference frame, and the deformations measured relative to this frame, are all inertially decoupled.

The price to be paid for the use of mean axes is very steep, however, because the constraints defining these axes are not easy to enforce. For this reason, it is common to invoke the use of mean axes to justify inertial decoupling without enforcing the constraints. In the case of flexible aircraft, even when the constraints are satisfied, the benefits of inertial decoupling are highly questionable because the equations of motion remain coupled through the aerodynamic forces. Moreover, if one insists on using mean axes, then the aerodynamic forces must also be expressed in terms of components along the same mean axes, which is a very tedious task at best. The use of mean axes without satisfying the constraints on the motion variables and without expressing the aerodynamic forces in terms of components along the mean axes is common practice in flexible aircraft dynamics, which raises serious questions as to the validity of the corresponding results.

The subject of aircraft dynamics is very broad, and the approach to the problems depends on the type of aircraft under consideration, its mission, and its dynamic characteristics. Traditionally, the subject has been divided into flight dynamics and aeroelasticity. Flight dynamics is concerned for the most part with rigid aircraft undergoing given maneuvers. On the other hand, aeroelasticity is concerned mainly with nonmaneuvering flexible aircraft. In fact, one of the most commonly used model consists of a flexible wing fixed at the root. Although flight dynamics and aeroelasticity have been developed as separate disciplines, the need for considering interacting effects was recognized quite early.[1–3] Still, relatively few attempts have been made to link the two disciplines and, when such attempts were made, almost invariably the scope was quite limited. This lack of interest in linking flight dynamics and aeroelasticity can be attributed to a reluctance to increase the complexity of the problem to a significant extent at a time when powerful computers capable of solving such problems were not available. As a result, problems combining flight dynamics and aeroelasticity effects have tended to be subjected to many simplifying assumptions designed to permit largely analytical solutions. In one of the first references on the subject, Bisplinghoff and Ashley[4] derived scalar equations of motion for an unrestrained flexible vehicle. The equations consisted of three inertially decoupled sets, one for the rigid-body translations, one for the rigid-body rotations, and one for the elastic deformations. An integrated analytical treatment of the equilibrium and stability of flexible aircraft was presented by Milne.[5] In Part I, he derived linearized equations of motion about a steady state, assuming small elastic deformations and rigid-body translations and rotations. Although the constraint equations defining the mean axes were given, the formulation seems to have used body axes attached to the undeformed aircraft. In Part II, the general analysis was applied to the study of equilibrium and longitudinal stability about equilibrium of an aircraft having longitudinal flexibility only. A monograph by Taylor and Woodcock[6] consists of two parts representing different approaches to the same problem. In Part I, Taylor presents a very lucid summary of the equations of motion for deformable aircraft derived by Bisplinghoff and Ashley[4] and by Milne.[5] In Part II, Woodcock uses an unorthodox form of Lagrange's equations to derive scalar perturbation equations of motion about a given "datum motion," not necessarily corresponding to steady level rectilinear flight; the equations are in terms of body-fixed axes. The question of aerodynamics receives scant attention in both parts. An extensive report by Dusto et al.,[7] resulting in a computer program known as FLEXSTAB, integrates flexible body mechanics with a low-frequency aerodynamics. The flexible aircraft mechanics uses free vibration modes superimposed on rigid-body dynamics. The equations are expressed in terms of steady perturbations about a reference motion to determine dynamic stability by characteristic roots or by time histories following an initial perturbation or some gust disturbance. There are three major concerns: The first concern is that the structural dynamics formulation is in terms of mean axes. The second concern is that the aerodynamics is in terms of a different set of axes, namely, "fluid axes." The third concern is the relatively long time required to run FLEXSTAB. Inconsistencies in control configured vehicles are highlighted by Schwanz,[8] who suggests that familiarity of flight control specialists with aerodynamics, structures, modern dynamics, and control can minimize and perhaps avoid these inconsistencies. Weisshaar and Zeiler[9] discuss the importance of including aircraft rigid-body modes in the aeroelastic analysis of forward swept wing aircraft. They show that "body-freedom" flutter and aircraft aeroelastic divergence, not wing divergence, are the primary aeroelastic instabilities. Cerra et al.[10] developed a linear model of an elastic aircraft providing the capability of analyzing the coupling between the rigid-body motions and the elastic motions. The model can be used for stability and control analyses. Using Lagrange's equations, Waszak and Schmidt[11] derived the equations of motion for a flexible aircraft. The strip theory was used to obtain closed-form integral expressions for the generalized aerodynamic forces. Moreover, the use of mean axes permitted inertial decoupling of the rigid-body translations, rigid-body rotations, and elastic deformations, the latter being expressed in terms of aircraft vibration modes. The modeling method was applied to a generic elastic aircraft, and the model was used for a parametric study of the flexibility effects.

The equations of motion in Refs. 3–11 were derived either by means of Newtonian mechanics or by means of standard Lagrange's equations. These approaches are more suitable when the motions are expressed in terms of inertial axes and/or when the rotations are in terms of Euler's angles. Yet, in the case of aircraft, it is more convenient to express the motion in terms of components along body axes. This is common practice in flight dynamics, in which case the angular velocities in terms of body axes are the well-known roll, pitch, and yaw. Of course, equations in terms of inertial axes and/or Eulerian angles can always be transformed into equations in terms of body axes through coordinate transformations. It is appreciably simpler, however, to derive the equations of motion directly in terms of body axes, which can be done through the use of Lagrange's equations in terms of quasi-coordinates.[12]

Motivated by problems in dynamics of spacecraft with flexible appendages, Meirovitch and Nelson[13] derived for the first time hybrid (ordinary and partial) differential equations of motion coupling rigid-body rotations and elastic deformations. The elastic deformations were measured relative to a set of body axes attached to the undeformed spacecraft and the rotational motions were in terms of quasi-coordinates. The explicit formulation of Ref. 13 was extended by Meirovitch[14] to a generic whole flexible body by deriving a set of hybrid equations of motion in terms of quasi-coordinates, treating for the first time translational velocities as quasi-velocities; the equations were then cast in compact state form. The vector-matrix formulation of Ref. 14 has the important practical advantage that it permits efficient computer solutions, as shown later in this paper. The developments of Ref. 14 were extended by Meirovitch[15] and Meirovitch and Stemple[16] to flexible multibody systems. Then, the approach of Refs. 14–16 was used by Meirovitch[17] to produce a definitive unified theory for the dynamics of flexible aircraft. Generic state equations describing the flight of flexible aircraft were first derived in hybrid form and subsequently discretized in space. Then, using a perturbation approach, the discrete state equations were separated into a set of nonlinear flight dynamics equations for the rigid-body variables and a set of linear extended aeroelasticity equations (see later definition) for the elastic variables and perturbations in the rigid-body variables. Nydick and Friedmann[18] applied the equations of motion in terms of quasi-coordinates derived in Ref. 14 to the analysis of a hypersonic vehicle in free flight. The nonlinear equations were linearized about a trim state obtained by using a rigid-body trim model and steady hypersonic aerodynamics. Flutter derivatives were calculated by means of piston theory. Although the formulation uses mean axes, there is no attempt to enforce the associated constraint equations, nor to express the aerodynamic forces in terms of mean axes components, so that the results are questionable. Winther et al.[19] provide an efficient procedure for real-time simulations, including transformations to a body-axes coordinate system and integration of the structural dynamic equations with the quasi-steady, nonlinear equations of motion. The generic formulation of Ref. 17 was used by Meirovitch and Tuzcu[20] to derive explicit equations of motion in terms of quasi-coordinates for a flexible aircraft and to cast the equations in a special state form suitable for simulation on a computer. Because of the relative ease of integration into the unified formulation and to the computational speed advantages, the aerodynamic forces were derived by means of strip theory. An approach entirely different from that in Ref. 20 is proposed by Fornasier et al.[21] Indeed, Ref. 21 is concerned essentially with the fluid–structure interaction in a flexible aircraft. To this end, it uses "temporal and spatial algorithms" to make two independently developed computer codes, one for computational fluid dynamics and one for computational structural mechanics, work together. The scope of Ref. 21 is relatively limited because the aircraft is assumed to follow a known preset trajectory, so that there are no rigid-body degrees of freedom, and there are no controls. The computations require a great deal of time, as witnessed by the fact that several 5-s simulations, including some of the wing tip displacement, took approximately 35 h on a 32-processor SGI computer.

This paper develops a dynamic formulation capable of simulating on a computer the response of flying flexible aircraft to external stimuli. It integrates seamlessly in a single and consistent mathematical formulation all the necessary material from the pertinent disciplines, namely, analytical dynamics, structural dynamics, aerodynamics, and controls. The formulation is modular in nature, which implies that any structural modeling, aerodynamic theory, and control method can be used, provided that they lend themselves to seamless integration, as explained later; only the analytical dynamics framework remains the same. The unified formulation is based on fundamental principles and incorporates in a natural manner both the rigid-body motions and the elastic deformations, and the couplings thereof, as well as the aerodynamic, propulsion, control, and gravity forces. It is convenient to regard the aircraft as a collection of flexible components acting together as a single system, where the components are identified broadly as the fuselage, wing, and empennage.

The choice in this paper is to work with a reference frame attached to the undeformed aircraft, which does not have the drawbacks of the mean axes. However, because the elastic deformations prevent the origin of such a frame from coinciding with the mass center and the axes themselves from coinciding with the principal axes for all times, there is no preferred choice of a reference frame; we base the choice on geometric considerations. In particular, we attach a set of body axes to the undeformed fuselage, where one of the axes is along the symmetry axis. For convenience, sets of body axes are also attached to the other flexible components, such as the wing and the empennage. Ultimately, however, all motions are referred to the fuselage body axes, which act as a reference frame for the whole aircraft.

The formulation represents a new and comprehensive theory for the dynamics and control of maneuvering flexible aircraft. The theory transcends flight dynamics and aeroelasticity, and in fact it contains them as special cases. The mathematical formulation is based on equations of motion in terms of quasi-coordinates derived first for flexible spacecraft[14] and later adapted to flexible aircraft.[17,20] The formulation is hybrid in nature, in the sense that it consists of ordinary differential equations for the rigid-body translations and rotations of the aircraft as a whole and partial differential equations for the elastic deformations of the flexible components of the aircraft, namely, the fuselage, wing, and empennage. For practical reasons, the partial differential equations for the individual components are discretized in space, obtaining a relatively large set of second-order ordinary differential equations for the whole aircraft. The simulation of the aircraft response requires the integration of the differential equations of motion, which makes it necessary to transform the set of second-order differential equations into a set of first-order differential equations, namely, into a set of state equations.

Because of various nonlinearities involved, such as those due to rigid-body motions and aerodynamic forces, the integration of the state equations must be carried out numerically on a computer, which can only be done in discrete time. In some cases, the integration of the state equations must be carried out in real time. If the dynamic characteristics are such that the time step must be relatively short, perhaps of the order of 0.01 s, most aerodynamic theories in current use must be ruled out because the computation of the dynamic pressure over the entire aircraft is sure to take considerably longer than that. Hence, a new method for computing the dynamic pressure must be developed, one characterized by high computational speed, even if some accuracy must be sacrificed. Moreover, the method for computing the dynamic pressure must lend itself to ready integration into the overall formulation. If the formulation is to be used for aircraft design, then real-time simulation may not really be necessary, although online simulation may. However, the size of the sampling period, which is determined by the system dynamic characteristics, remains the same regardless of whether the simulation is in real time or only online. This demonstrates the need for a method for computing the dynamic pressure in a very short time. In this regard, a reasonably accurate approximate method may be acceptable.

The development of a new aerodynamic theory such as the one just described is likely to require a great deal of time, measured in many years. Until such theory becomes available, it is highly desirable to demonstrate how the unifying process works, which implies the use of an existing theory. One theory lending itself to ready integration is strip theory, which is capable of expressing the aerodynamic forces in terms of components along the aircraft body axes and describing them by using the same rigid-body and elastic variables used to describe the aircraft motions. Note that, because of the modularity of the formulation, any other aerodynamic theory can be substituted for strip theory, provided it can be readily integrated into the unified process and permits fast computation of the aerodynamic forces.

As indicated earlier, the equations of motion for flying flexible aircraft are nonlinear, where the nonlinearity is due to the rigid-body motions and the aerodynamic forces. Moreover, the equations tend to be of high order, the order depending on the discretization procedure employed. Hence, one can expect difficulties both with a stability analysis and with control design. In addition, difficulties can be experienced in the integration of the equations of motion because some of the variables describing the aircraft rigid-body motions tend to be large and the variables describing the elastic displacements tend to be small. Fortunately, the integration of the high-order, nonlinear set of differential equations is not necessary, or even desirable, a statement that can be justified by considering how aircraft are flown. Indeed, a pilot first chooses a flight path, consisting of the aircraft translations and rotations, and then sets the controls, i.e., the engines throttles and control surfaces angles, so as to realize the chosen path. Then, any deviations from the flight path, consisting of both elastic vibration and perturbations in the rigid-body variables, can be controlled by feedback control. This points to a perturbation approach as a natural way of treating the problem. More specifically, the solution can be represented as the sum of a zero-order part for the large rigid-body variables and a first-order part for the small elastic variables and perturbations in the rigid-body variables, where the zero-order quantities are larger than the first-order quantities by at least one order of magnitude. Then, the equations of motion can be separated into a zero-order problem for the aircraft rigid-body motions alone and a first-order problem for the elastic displacements and the perturbations in the rigid-body motions. The state equations for the zero-order problem can be identified as the equations of flight dynamics and can be used to describe aircraft maneuvers. On the other hand, the state equations for the first-order problem describe extended perturbation equations, which are considerably broader in scope than the commonly encountered aeroservoelasticity equations. Indeed, the extended perturbation problem is characterized by the following: 1) It is concerned with whole aircraft, and not merely with wings. 2) All six rigid-body degrees of freedom are automatically included. 3) The motions are expressed in terms of components along noninertial reference frames to accommodate all possible aircraft maneuvers. 4) The perturbation equations are subject to inputs from the aircraft maneuvers, so that there is one set of perturbation equations for every conceivable maneuver. 5) The task of ensuring that the aircraft flies along the intended path is carried out by feedback control in conjunction with a stochastic state estimator.

The flight dynamics equations are in general nonlinear and describe the translations and rotations of the aircraft as if it were rigid. They can be used to design given maneuvers of an aircraft, which amounts to solving an inverse dynamics problem. In the commonly encountered direct problems in dynamics of rigid bodies, the forces are given and the equations of motion are solved for the state, i.e., for the positions and velocities. In the context of the present formulation, however, the state representing a desired maneuver is given, and the problem amounts to determining the engines throttles and the control surfaces angles permitting the realization of the maneuver; this represents an inverse dynamics problem. On the other hand, the extended perturbation equations are linear, but they contain the state and forces from the flight dynamics problem as coefficients and as an input. If the flight dynamics problem represents steady level cruise or a steady level turn maneuver, then the zero-order state and forces are constant, and the system of extended perturbation equations is linear time-invariant. In this case, the state equations lend themselves to a standard stability check, such as one based on the roots of the eigenvalue problem, to control design by commonly used techniques, such as the linear quadratic Gaussian (LQG) method, and to ready integration for simulation of the aircraft response to external stimuli. If the flight dynamics problem represents a time-dependent maneuver, such as the transition from one steady state to another, then the zero-order state and forces depend on time, and the extended perturbation state equations are linear time-varying, which precludes a standard stability analysis. However, the state equations still permit control design and response simulation, albeit numerically in discrete time.

The paper contains a numerical example for a model of a flexible aircraft containing 76 states, 12 rigid-body states and 64 elastic states. Two cases are considered, one corresponding to steady level cruise and the other to a level steady turn maneuver. The extended perturbation problems are derived and used to design feedback controls guaranteeing the vanishing of the rigid-body perturbations and the elastic vibration, thus ensuring the stability of the maneuver and the comfort of the flight. The control design consists of a linear quadratic regulator in conjunction with a stochastic observer (LQG). The integrated process is demonstrated by means of a numerical example including a variety of rigid-body and elastic displacements time simulations, together with the corresponding control time histories, all carried out on a 1-GHz personal computer.

The developments presented here should be regarded as a first, but significant step toward establishing a new direction in the dynamics and control of maneuvering flexible aircraft. The expectation is that they will give new impetus to research in aeronautics, thus providing a useful tool in the design of new generations of aircraft, such as autonomous unmanned aerial vehicles (UAVs).

II. Hybrid Equations of Motion in Terms of Quasi-Coordinates

We regard the aircraft model shown in Fig. 1 as a flexible multibody system subjected to gravity, aerodynamic, propulsion, and control forces, where the bodies can be broadly identified as the fuselage, wing, and empennage. The motion of the aircraft can be conveniently described by attaching a reference frame $x_f y_f z_f$ to the undeformed fuselage (Fig. 1), as well as corresponding reference frames $x_w y_w z_w$ and $x_e y_e z_e$ attached to the wing and empennage, respectively, where the various reference frames represent respective body axes. Then, the motion can be described by six rigid-body degrees of freedom of the fuselage body axes, three translations and three rotations, and by the elastic deformation of every point of each flexible component relative to the respective body axes.

From Ref. 17, with provisions made for members in torsion, as well as for damping, the hybrid equations of motion for the whole

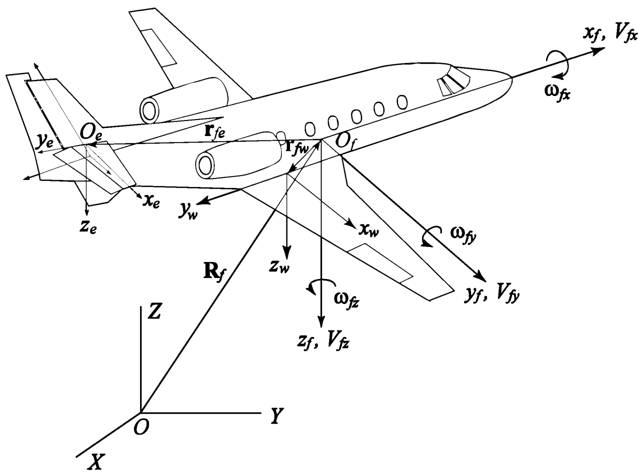

Fig. 1 Flexible aircraft model.

flexible aircraft in terms of quasi-coordinates have the generic form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{V}_f}\right) + \tilde{\omega}_f \frac{\partial L}{\partial \boldsymbol{V}_f} - C_f \frac{\partial L}{\partial \boldsymbol{R}_f} = \boldsymbol{F}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{\omega}_f}\right) + \tilde{V}_f \frac{\partial L}{\partial \boldsymbol{V}_f} + \tilde{\omega}_f \frac{\partial L}{\partial \boldsymbol{\omega}_f} - \left(E_f^T\right)^{-1} \frac{\partial L}{\partial \boldsymbol{\theta}_f} = \boldsymbol{M}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{v}_i}\right) - \frac{\partial \hat{L}_i}{\partial \boldsymbol{u}_i} + \frac{\partial \hat{\mathcal{F}}_{ui}}{\partial \dot{\boldsymbol{u}}_i} + \mathcal{L}_{ui}\boldsymbol{u}_i = \hat{\boldsymbol{U}}_i$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\alpha}_i}\right) + \frac{\partial \hat{\mathcal{F}}_{\alpha i}}{\partial \dot{\boldsymbol{\psi}}_i} + \mathcal{L}_{\psi i}\boldsymbol{\psi}_i = \hat{\boldsymbol{\Psi}}_i \tag{1}$$

where $i = f$ (fuselage), w (wing), and e (empennage).

Under the assumption that axes $x_f y_f z_f$ are obtained from axes XYZ through the sequence of rotations ψ about Z to axes $x_1 y_1 z_1$, θ about y_1 to $x_2 y_2 z_2$, and ϕ about x_2 to $x_f y_f z_f$, the matrices C_f and E_f can be obtained from Ref. 22. The elastic displacement vectors $\boldsymbol{u}_i$ and $\boldsymbol{\psi}_i$ are subject to given boundary conditions at the interface between bodies. Equations (1) involve the Lagrangian $L = T - V$, the Rayleigh dissipation function densities $\hat{\mathcal{F}}_{ui}$ and $\hat{\mathcal{F}}_{\psi i}$, and the stiffness operators $\mathcal{L}_{ui}$ and $\mathcal{L}_{\psi i}$. The kinetic energy for the whole aircraft can be written as

$$T = \sum_{i=1}^{3} T_i = \sum_{i=1}^{3} \frac{1}{2} \int \bar{\boldsymbol{V}}_i^T \bar{\boldsymbol{V}}_i \,\mathrm{d}m_i \tag{2}$$

in which $\bar{\boldsymbol{V}}_i$ are velocity vectors of typical points in the components and $\mathrm{d}m_i$ are corresponding mass differential elements. The velocity of a point in the fuselage can be written as

$$\bar{\boldsymbol{V}}_f(\boldsymbol{r}_f, t) = \boldsymbol{V}_f(t) + [\tilde{r}_f + \tilde{u}_f(\boldsymbol{r}_f, t)]^T[\boldsymbol{\omega}_f(t) + \boldsymbol{\alpha}_f(\boldsymbol{r}_f, t)]$$
$$+ \boldsymbol{v}_f(\boldsymbol{r}_f, t) \cong \boldsymbol{V}_f + (\tilde{r}_f + \tilde{u}_f)^T \boldsymbol{\omega}_f + \tilde{r}_f^T \boldsymbol{\alpha}_f + \boldsymbol{v}_f \tag{3}$$

where $\tilde{r}_f$ and $\tilde{u}_f$ are skew symmetric matrices[12] corresponding to $\boldsymbol{r}_f$ and $\boldsymbol{u}_f$. Moreover, the velocity of a point on the wing has the expression

$$\bar{\boldsymbol{V}}_w(\boldsymbol{r}_w, t) \cong C_w \boldsymbol{V}_f + \left[C_w(\tilde{r}_{fw} + \tilde{u}_{fw})^T + (\tilde{r}_w + \tilde{u}_w)^T C_w\right]\boldsymbol{\omega}_f$$
$$+ \tilde{r}_w^T C_w(\boldsymbol{\Omega}_{fw} + \boldsymbol{\alpha}_{fw}) + C_w\left(\boldsymbol{v}_{fw} + \tilde{r}_{fw}^T \boldsymbol{\alpha}_{fw}\right) + \tilde{r}_w^T \boldsymbol{\alpha}_w + \boldsymbol{v}_w \tag{4}$$

in which $\boldsymbol{\Omega}_{fw} = [0 \;\; -\partial \dot{u}_{fz}/\partial x_f \;\; \partial \dot{u}_{fy}/\partial x_f]|_{\boldsymbol{r}_{fw}}^T$ is the angular velocity of the fuselage at $\boldsymbol{r}_{fw}$ due to bending and $\boldsymbol{\alpha}_{fw} = [\alpha_{fw} \;\; 0 \;\; 0]|_{\boldsymbol{r}_{fw}}^T$ is the angular velocity of the fuselage at $\boldsymbol{r}_{fw}$ due to torsion. The velocity $\bar{\boldsymbol{V}}_e(\boldsymbol{r}_e, t)$ of a point on the empennage can be obtained from Eq. (4) by simply replacing w by e.

The potential energy can be expressed in terms of the operators $\mathcal{L}_{ui}$, $\mathcal{L}_{\psi i}$ $(i = f, w, e)$, but it is more conveniently expressed as the strain energy, as shown in the Numerical Example section. Moreover, the Rayleigh dissipation function densities can be expressed in the form

$$\hat{\mathcal{F}}_{ui} = \frac{1}{2} c_{ui} E I_i \frac{\partial^2 \dot{\boldsymbol{u}}_i^T}{\partial x_i^2} \frac{\partial^2 \dot{\boldsymbol{u}}_i}{\partial x_i^2}, \qquad \hat{\mathcal{F}}_{\psi i} = \frac{1}{2} c_{\psi i} G J_i \frac{\partial \dot{\boldsymbol{\psi}}_i^T}{\partial x_i} \frac{\partial \dot{\boldsymbol{\psi}}_i}{\partial x_i}$$
$$i = f, w, e \tag{5}$$

The kinetic energy T, the potential energy V, the dissipation functions $\hat{\mathcal{F}}_{ui}$ and $\hat{\mathcal{F}}_{\psi i}$, and the operators $\mathcal{L}_{ui}$ and $\mathcal{L}_{\psi i}$, when inserted into Eqs. (1), permit the derivation of explicit hybrid equations of motion. For practical reasons, we do not pursue hybrid equations any further, and approximate instead the partial differential equations by sets of ordinary differential equations.

III. Spatial Discretization of the Distributed Variables

For the most part, aircraft are modeled as discrete systems, either by regarding the inertia and stiffness properties as lumped from the onset or by spatial discretization. We consider spatial discretization of the individual aircraft components separately. To this end, we can use either the Galerkin method or the finite element method (see Ref. 23) and introduce the expansions

$$\boldsymbol{u}_i(\boldsymbol{r}_i, t) = \Phi_{ui}(\boldsymbol{r}_i)\boldsymbol{q}_{ui}(t), \qquad \boldsymbol{\psi}_i(\boldsymbol{r}_i, t) = \Phi_{\psi i}(\boldsymbol{r}_i)\boldsymbol{q}_{\psi i}(t)$$
$$i = f, w, e \tag{6}$$

where Φ_{ui} and $\Phi_{\psi i}$ are matrices of component shape functions and $\boldsymbol{q}_{ui}$ and $\boldsymbol{q}_{\psi i}$ are corersponding vectors of generalized coordinates. Some guidelines concerning the choice of shape functions can be found in Ref. 23. Moreover, we denote the associated generalized velocities by $\boldsymbol{s}_{ui}(t) = \dot{\boldsymbol{q}}_{ui}(t)$, $\boldsymbol{s}_{\psi i}(t) = \dot{\boldsymbol{q}}_{\psi i}(t)$. In anticipation of later needs, we write the velocity vectors for points on the individual components in the two discrete forms

$$\bar{\boldsymbol{V}}_f(\boldsymbol{r}_f, t) = \boldsymbol{V}_f + (\tilde{r}_f + \widetilde{\Phi_{uf}\boldsymbol{q}_{uf}})^T \boldsymbol{\omega}_f + \Phi_{uf}\boldsymbol{s}_{uf} + \tilde{r}_f^T \Phi_{\psi f}\boldsymbol{s}_{\psi f}$$
$$= \boldsymbol{V}_f + \tilde{r}_f^T \boldsymbol{\omega}_f + \tilde{\omega}_f \Phi_{uf}\boldsymbol{q}_{uf} + \Phi_{uf}\boldsymbol{s}_{uf} + \tilde{r}_f^T \Phi_{\psi f}\boldsymbol{s}_{\psi f}$$

$$\bar{\boldsymbol{V}}_w(\boldsymbol{r}_w, t) = C_w \boldsymbol{V}_f + \big[C_w(\tilde{r}_{fw} + \widetilde{\Phi_{ufw}\boldsymbol{q}_{uf}})^T$$
$$+ (\tilde{r}_w + \widetilde{\Phi_{uw}\boldsymbol{q}_{uw}})^T C_w\big]\boldsymbol{\omega}_f$$
$$+ \left(\tilde{r}_w^T C_w \Delta \Phi_{ufw} + C_w \Phi_{ufw}\right)\boldsymbol{s}_{uf} + \Phi_{uw}\boldsymbol{s}_{uw}$$
$$+ \left(\tilde{r}_w^T C_w \Phi_{\psi fw} + C_w \tilde{r}_{fw}^T \Phi_{\psi fw}\right)\boldsymbol{s}_{\psi f} + \tilde{r}_w^T \Phi_{\psi w}\boldsymbol{s}_{\psi w}$$
$$= C_w \boldsymbol{V}_f + \left(C_w \tilde{r}_{fw}^T + \tilde{r}_w^T C_w\right)\boldsymbol{\omega}_f + C_w \tilde{\omega}_f \Phi_{ufw}\boldsymbol{q}_{uf}$$
$$+ \widetilde{C_w \boldsymbol{\omega}_f}\Phi_{uw}\boldsymbol{q}_{uw} + \left(\tilde{r}_w^T C_w \Delta \Phi_{ufw} + C_w \Phi_{ufw}\right)\boldsymbol{s}_{uf}$$
$$+ \Phi_{uw}\boldsymbol{s}_{uw} + \left(\tilde{r}_w^T C_w \Phi_{\psi fw} + C_w \tilde{r}_{fw}^T \Phi_{\psi fw}\right)\boldsymbol{s}_{\psi f}$$
$$+ \tilde{r}_w^T \Phi_{\psi w}\boldsymbol{s}_{\psi w} \tag{7}$$

in which

$$\Delta = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{\partial}{\partial x_f} \\ \dfrac{\partial}{\partial x_f} & 0 \end{bmatrix}, \qquad \Phi_{uf} = \begin{bmatrix} \boldsymbol{0}^T & \boldsymbol{0}^T \\ \phi_{ufy}^T & \boldsymbol{0}^T \\ \boldsymbol{0}^T & \phi_{ufz}^T \end{bmatrix}$$

$$\Phi_{ufw} = \Phi_{uf}(\boldsymbol{r}_{fw}) \tag{8}$$

Expressions analogous to Φ_{uf} can be written for $\Phi_{\psi f}$ and $\Phi_{\psi fw}$. Moreover, $\bar{\boldsymbol{V}}_e(\boldsymbol{r}_e, t)$ can be obtained from $\bar{\boldsymbol{V}}_w(\boldsymbol{r}_w, t)$ by simply replacing w by e. Note that the tilde over $\Phi_{uf}\boldsymbol{q}_{uf}$ indicates a skew symmetric matrix associated with the vector,[12] etc. Inserting the first forms of Eqs. (7) into Eq. (2) and carrying out the indicated operations, we can write the kinetic energy in the compact form

$$T=\frac{1}{2}\int \bar{\boldsymbol{V}}_f^T\bar{\boldsymbol{V}}_f\,\mathrm{d}m_f+\frac{1}{2}\int \bar{\boldsymbol{V}}_w^T\bar{\boldsymbol{V}}_w\,\mathrm{d}m_w+\frac{1}{2}\int \bar{\boldsymbol{V}}_e^T\bar{\boldsymbol{V}}_e\,\mathrm{d}m_e$$
$$=\frac{1}{2}\boldsymbol{V}^T M\boldsymbol{V} \tag{9}$$

where $\boldsymbol{V}=[\boldsymbol{V}_f^T\ \boldsymbol{\omega}_f^T\ \boldsymbol{s}_{uf}^T\ \boldsymbol{s}_{uw}^T\ \boldsymbol{s}_{ue}^T\ \boldsymbol{s}_{\psi f}^T\ \boldsymbol{s}_{\psi w}^T\ \boldsymbol{s}_{\psi e}^T]^T=[\boldsymbol{V}_1^T\ \boldsymbol{V}_2^T\cdots\boldsymbol{V}_8^T]^T$ is the discrete system velocity vector and M is the system mass matrix, a symmetric matrix that can be expressed in partitioned form with the submatrices M_{ij}, $i,j=1,2,\ldots,8$. As a matter of interest, we note that $M_{11}=mI$, $M_{12}=\tilde{S}^T$, and $M_{22}=J$. All the submatrices are given in Ref. 24.

In a similar fashion, we insert Eqs. (6) into Eqs. (5), integrate over the respective components domains, and obtain the generalized Rayleigh's dissipation functions

$$\mathcal{F}_{ui}=\int_{D_i}\hat{\mathcal{F}}_{ui}\,\mathrm{d}D_i=\frac{1}{2}\int_{D_i}c_{ui}EI_i\dot{\boldsymbol{q}}_{ui}^T\frac{\mathrm{d}^2\Phi_{ui}^T}{\mathrm{d}x_i^2}\frac{\mathrm{d}^2\Phi_{ui}}{\mathrm{d}x_i^2}\dot{\boldsymbol{q}}_{ui}\,\mathrm{d}D_i$$
$$=\frac{1}{2}\dot{\boldsymbol{q}}_{ui}^TC_{ui}\dot{\boldsymbol{q}}_{ui}$$
$$\mathcal{F}_{\psi i}=\int_{D_i}\hat{\mathcal{F}}_{\psi i}\,\mathrm{d}D_i=\frac{1}{2}\int_{D_i}c_{\psi i}GJ_i\dot{\boldsymbol{q}}_{\psi i}^T\frac{\mathrm{d}\Phi_{\psi i}^T}{\mathrm{d}x_i}\frac{\mathrm{d}\Phi_{\psi i}}{\mathrm{d}x_i}\dot{\boldsymbol{q}}_{\psi i}\,\mathrm{d}D_i$$
$$=\frac{1}{2}\dot{\boldsymbol{q}}_{\psi i}^TC_{\psi i}\dot{\boldsymbol{q}}_{\psi i},\qquad i=f,w,e \tag{10}$$

where

$$C_{ui}=\int_{D_i}c_{ui}EI_i\frac{\mathrm{d}^2\Phi_{ui}^T}{\mathrm{d}x_i^2}\frac{\mathrm{d}^2\Phi_{ui}}{\mathrm{d}x_i^2}\,\mathrm{d}D_i$$
$$C_{\psi i}=\int_{D_i}c_{\psi i}GJ_i\frac{\mathrm{d}\Phi_{\psi i}^T}{\mathrm{d}x_i}\frac{\mathrm{d}\Phi_{\psi i}}{\mathrm{d}x_i}\,\mathrm{d}D_i,\qquad i=f,w,e \tag{11}$$

are damping matrices.

Next, we denote the momentum vector for the whole aircraft by $\boldsymbol{p}=[\boldsymbol{p}_{Vf}^T\ \boldsymbol{p}_{\omega f}^T\ \boldsymbol{p}_{uf}^T\ \boldsymbol{p}_{uw}^T\ \boldsymbol{p}_{ue}^T\ \boldsymbol{p}_{\psi f}^T\ \boldsymbol{p}_{\psi w}^T\ \boldsymbol{p}_{\psi e}^T]^T=[\boldsymbol{p}_1^T\ \boldsymbol{p}_2^T\cdots\boldsymbol{p}_8^T]^T$, so that we can write

$$\boldsymbol{p}=\frac{\partial T}{\partial \boldsymbol{V}}=M\boldsymbol{V} \tag{12}$$

where the individual momenta are given by

$$\boldsymbol{p}_{Vf}=\frac{\partial T}{\partial \boldsymbol{V}_f}=\boldsymbol{p}_1=\sum_{j=1}^{8}M_{1j}\boldsymbol{V}_j,\ \boldsymbol{p}_{\omega f}=\frac{\partial T}{\partial \boldsymbol{\omega}_f}=\boldsymbol{p}_2=\sum_{j=1}^{8}M_{2j}\boldsymbol{V}_j$$
$$\boldsymbol{p}_{uf}=\frac{\partial T}{\partial \boldsymbol{s}_{uf}}=\boldsymbol{p}_3=\sum_{j=1}^{8}M_{3j}\boldsymbol{V}_j,\ldots,\boldsymbol{p}_{\psi e}=\frac{\partial T}{\partial \boldsymbol{s}_{\psi e}}=\boldsymbol{p}_8=\sum_{j=1}^{8}M_{8j}\boldsymbol{V}_j \tag{13}$$

Finally, adding some obvious kinematical identities[24] to the discretized version of Eqs. (1), the state equations can be written in the special form

$$\dot{\boldsymbol{R}}_f=C_f^T\boldsymbol{V}_f,\ \ \dot{\boldsymbol{\theta}}_f=E_f^{-1}\boldsymbol{\omega}_f;\ \ \dot{\boldsymbol{q}}_{ui}=\boldsymbol{s}_{ui},\ \ \dot{\boldsymbol{q}}_{\psi i}=\boldsymbol{s}_{\psi i},\ \ i=f,w,e$$
$$\dot{\boldsymbol{p}}_{Vf}=-\tilde{\omega}_f\boldsymbol{p}_{Vf}+\boldsymbol{F},\qquad \dot{\boldsymbol{p}}_{\omega f}=-\tilde{V}_f\boldsymbol{p}_{Vf}-\tilde{\omega}_f\boldsymbol{p}_{\omega f}+\boldsymbol{M}$$
$$\left.\begin{aligned}\dot{\boldsymbol{p}}_{ui}&=\frac{\partial T}{\partial \boldsymbol{q}_{ui}}-K_{ui}\boldsymbol{q}_{ui}-C_{ui}\boldsymbol{s}_{ui}+\boldsymbol{Q}_{ui}\\ \dot{\boldsymbol{p}}_{\psi i}&=-K_{\psi i}\boldsymbol{q}_{\psi i}-C_{\psi i}\boldsymbol{s}_{\psi i}+\boldsymbol{Q}_{\psi i}\end{aligned}\right\}\quad i=f,w,e \tag{14}$$

where, using Eq. (2) in conjunction with the second form of $\bar{\boldsymbol{V}}_f$, $\bar{\boldsymbol{V}}_w$, and $\bar{\boldsymbol{V}}_e$, Eqs. (7),

$$\frac{\partial T}{\partial \boldsymbol{q}_{uf}}=\int \Phi_{uf}^T\tilde{\omega}_f^T\bar{\boldsymbol{V}}_f\,\mathrm{d}m_f+\Phi_{ufw}^T\tilde{\omega}_f^TC_w^T\int\bar{\boldsymbol{V}}_w\,\mathrm{d}m_w$$
$$+\Phi_{ufe}^T\tilde{\omega}_f^TC_e^T\int\bar{\boldsymbol{V}}_e\,\mathrm{d}m_e$$
$$\frac{\partial T}{\partial \boldsymbol{q}_{uw}}=\Phi_{uw}^T\widetilde{C_w\boldsymbol{\omega}_f}^T\int\bar{\boldsymbol{V}}_w\,\mathrm{d}m_w,\qquad \frac{\partial T}{\partial \boldsymbol{q}_{ue}}=\Phi_{ue}^T\widetilde{C_e\boldsymbol{\omega}_f}^T\int\bar{\boldsymbol{V}}_e\,\mathrm{d}m_e \tag{15}$$

Moreover,

$$K_{ui}=\int\Phi_{ui}^T\mathcal{L}_{ui}\Phi_{ui}\,\mathrm{d}D_i,\qquad K_{\psi i}=\int\Phi_{\psi i}^T\mathcal{L}_{\psi i}\Phi_{\psi i}\,\mathrm{d}D_i$$
$$i=f,w,e \tag{16}$$

are component stiffness matrices. In practice, K_{ui} and $K_{\psi i}$, $i=f,w,e$, can be obtained with greater ease from the strain energy directly, as shown in the Numerical Example section. Note that the term special form is used in the sense that momenta are used as auxiliary variables instead of velocities.

The quantities $\boldsymbol{F}$, $\boldsymbol{M}$, $\boldsymbol{Q}_{ui}$, and $\boldsymbol{Q}_{\psi i}$, $i=f,w,e$, appearing in the state equations, Eqs. (14), represent generalized forces. They are related to the actual distributed forces $\boldsymbol{f}_i(\boldsymbol{r}_i,t)$ over components i due to gravity, aerodynamics, and controls and the engine thrust $\boldsymbol{F}_E\delta(\boldsymbol{r}-\boldsymbol{r}_E)$, in which $\boldsymbol{r}_E$ appearing in the spatial Dirac delta function[23] denotes the location of the engines. If some control forces are concentrated, they can also be treated as distributed, as in the case of the engine thrust. The relation between the generalized forces and the actual forces can be obtained by means of the virtual work, which can be expressed as

$$\delta\bar{W}=\sum_i\int_{D_i}\left[\boldsymbol{f}_i^T+\boldsymbol{F}_E^T\delta(\boldsymbol{r}-\boldsymbol{r}_E)\right]\delta\bar{\boldsymbol{R}}_i^*\,\mathrm{d}D_i \tag{17}$$

The vector $\delta\bar{\boldsymbol{R}}_i^*$ is related to the virtual quasi-displacement vectors corresponding to the quasi-velocities used to describe the motion of the aircraft components. Indeed, using Eqs. (7), we can write

$$\delta\bar{\boldsymbol{R}}_f^*=\delta\boldsymbol{R}_f^*+(\tilde{r}_f+\widetilde{\Phi_{uf}\boldsymbol{q}_{uf}})^T\delta\boldsymbol{\theta}_f^*+\Phi_{uf}\delta\boldsymbol{q}_{uf}+\tilde{r}_f^T\Phi_{\psi f}\delta\boldsymbol{q}_{\psi f}$$
$$\delta\bar{\boldsymbol{R}}_w^*=C_w\delta\boldsymbol{R}_f^*+\left[C_w(\tilde{r}_{fw}+\widetilde{\Phi_{ufw}\boldsymbol{q}_{uf}})^T+(\tilde{r}_w+\widetilde{\Phi_{uw}\boldsymbol{q}_{uw}})^TC_w\right]\delta\boldsymbol{\theta}_f^*$$
$$+\left(\tilde{r}_w^TC_w\Delta\Phi_{ufw}+C_w\Phi_{ufw}\right)\delta\boldsymbol{q}_{uf}+\Phi_{uw}\delta\boldsymbol{q}_{uw}$$
$$+\left(\tilde{r}_w^TC_w\Phi_{\psi fw}+C_w\tilde{r}_{fw}^T\Phi_{\psi fw}\right)\delta\boldsymbol{q}_{\psi f}+\tilde{r}_w^T\Phi_{\psi w}\delta\boldsymbol{q}_{\psi w} \tag{18}$$

and we note that $\delta\bar{\boldsymbol{R}}_e^*$ can be obtained from $\delta\bar{\boldsymbol{R}}_w^*$ by replacing w by e. Inserting Eqs. (18) into Eq. (17) and collecting terms, we can write the virtual work in terms of virtual generalized displacements, as follows:

$$\delta\bar{W}=\boldsymbol{F}^T\delta\boldsymbol{R}_f^*+\boldsymbol{M}^T\delta\boldsymbol{\theta}_f^*+\sum_i\left(\boldsymbol{Q}_{ui}^T\delta\boldsymbol{q}_{ui}+\boldsymbol{Q}_{\psi i}^T\delta\boldsymbol{q}_{\psi i}\right) \tag{19}$$

from which, assuming that the engines are mounted on the fuselage (Fig. 1), we obtain

$$\boldsymbol{F}=\int_{D_f}[\boldsymbol{f}_f+\boldsymbol{F}_E\delta(\boldsymbol{r}-\boldsymbol{r}_E)]\,\mathrm{d}D_f$$
$$+C_w^T\int_{D_w}\boldsymbol{f}_w\,\mathrm{d}D_w+C_e^T\int_{D_e}\boldsymbol{f}_e\,\mathrm{d}D_e$$
$$\boldsymbol{M}=\int_{D_f}(\tilde{r}_f+\widetilde{\Phi_{uf}\boldsymbol{q}_{uf}})[\boldsymbol{f}_f+\boldsymbol{F}_E\delta(\boldsymbol{r}-\boldsymbol{r}_E)]\,\mathrm{d}D_f$$
$$+\int_{D_w}\left[(\tilde{r}_{fw}+\widetilde{\Phi_{ufw}\boldsymbol{q}_{uf}})C_w^T+C_w^T(\tilde{r}_w+\widetilde{\Phi_{uw}\boldsymbol{q}_{uw}})\right]\boldsymbol{f}_w\,\mathrm{d}D_w$$
$$+\int_{D_e}\left[(\tilde{r}_{fe}+\widetilde{\Phi_{ufe}\boldsymbol{q}_{uf}})C_e^T+C_e^T(\tilde{r}_e+\widetilde{\Phi_{ue}\boldsymbol{q}_{ue}})\right]\boldsymbol{f}_e\,\mathrm{d}D_e$$

$$\begin{aligned}
\boldsymbol{Q}_{uf} = &\int_{D_f} \Phi_{uf}^T [\boldsymbol{f}_f + \boldsymbol{F}_E \delta(\boldsymbol{r} - \boldsymbol{r}_E)] \, \mathrm{d}D_f \\
&+ \int_{D_w} \left(\tilde{r}_w^T C_w \Delta\Phi_{ufw} + C_w \Phi_{ufw}\right)^T \boldsymbol{f}_w \, \mathrm{d}D_w \\
&+ \int_{D_e} \left(\tilde{r}_e^T C_e \Delta\Phi_{ufe} + C_e \Phi_{ufe}\right)^T \boldsymbol{f}_e \, \mathrm{d}D_e \\
\boldsymbol{Q}_{\psi f} = &\int_{D_f} \Phi_{\psi f}^T \tilde{r}_f [\boldsymbol{f}_f + \boldsymbol{F}_E \delta(\boldsymbol{r} - \boldsymbol{r}_E)] \, \mathrm{d}D_f \\
&+ \int_{D_w} \left(\tilde{r}_w^T C_w \Phi_{\psi fw} + C_w \tilde{r}_{fw}^T \Phi_{\psi fw}\right)^T \boldsymbol{f}_w \, \mathrm{d}D_w \\
&+ \int_{D_e} \left(\tilde{r}_e^T C_e \Phi_{\psi fe} + C_e \tilde{r}_{fe}^T \Phi_{\psi fe}\right)^T \boldsymbol{f}_e \, \mathrm{d}D_e \\
\boldsymbol{Q}_{ui} = &\int_{D_i} \Phi_{ui}^T \boldsymbol{f}_i \, \mathrm{d}D_i, \quad \boldsymbol{Q}_{\psi i} = \int_{D_i} \Phi_{\psi i}^T \tilde{r}_i \boldsymbol{f}_i \, \mathrm{d}D_i, \quad i = w, e
\end{aligned} \tag{20}$$

To complete the state equations, Eqs. (14), it is necessary to derive the stiffness matrices K_{ui} and $K_{\psi i}$, $i = f, w, e$, and the gravity, aerodynamic, and control forces.

The gravity forces per unit volume of components are simply

$$\boldsymbol{f}_{gf} = C_f \begin{bmatrix} 0 \\ 0 \\ \rho_f g \end{bmatrix}, \quad \boldsymbol{f}_{gw} = C_w C_f \begin{bmatrix} 0 \\ 0 \\ \rho_w g \end{bmatrix}, \quad \boldsymbol{f}_{ge} = C_e C_f \begin{bmatrix} 0 \\ 0 \\ \rho_e g \end{bmatrix} \tag{21}$$

The aerodynamic forces are the most difficult to integrate into the unified formulation, a statement that requires some elaboration. As pointed out in the Introduction, for seamless integration, the aerodynamic forces must be expressed in a form compatible with the present dynamic formulation. This implies that they must be expressed in terms of components along the same body axes as those used for the airframe and must be in terms of the same variables as those used to describe the rigid-body and elastic motions. This requirement is often ignored, particularly when mean axes are used as a reference frame. Moreover, for online, or real-time simulation of the aircraft response, the aerodynamic forces must lend themselves to very fast computation. Indeed, because such computations are carried out on a digital computer in discrete time, the aerodynamic forces, together with the other forces and the system state, must be updated every time step, typically a small fraction of a second. No existing whole aircraft aerodynamic theory is able to satisfy such stringent requirements, so that a suitable aerodynamic method must yet be developed. Such a method need not be unduly accurate in the presence of feedback controls because such controls, if they are robust, can tolerate small deviations from the actual aerodynamic forces. It is clear that such aerodynamics can only be developed with full consideration of the dynamic modeling of the aircraft, and its development is likely to demand a great deal of time. Fortunately, until such a theory becomes reality, it is still possible to demonstrate how aerodynamics fits into the integration process by using an existing theory satisfying the requirements just described, namely, strip theory.[11,25] The use of strip theory at this point has the added advantage of helping with the development of an eventual method by providing guidelines as to what is necessary for an aerodynamic method to be an integral part of the unified process. It should be stressed that the unified formulation developed here does not depend on any particular aerodynamic theory, and any other theory can be substituted for strip theory as long as it satisfies the outlined criteria.

From Ref. 24, the distributed aerodynamic force vectors acting on the fuselage are given by

$$\boldsymbol{f}_{af} = \begin{bmatrix} \ell_f \sin\alpha_f - d_f \cos\alpha_f \\ 0 \\ -\ell_f \cos\alpha_f - d_f \sin\alpha_f \end{bmatrix}, \qquad \boldsymbol{f}_{sf} = \begin{bmatrix} s_f \sin\beta_f \\ -s_f \cos\beta_f \\ 0 \end{bmatrix} \tag{22}$$

Similarly, the distributed aerodynamic forces on the wing and empennage are

$$\boldsymbol{f}_{aw} = \begin{bmatrix} 0 \\ \ell_w \sin\alpha_w - d_w \cos\alpha_w \\ -\ell_w \cos\alpha_w - d_w \sin\alpha_w \end{bmatrix}$$

$$\boldsymbol{f}_{ae} = \begin{bmatrix} 0 \\ \ell_e \sin\alpha_e - d_e \cos\alpha_e \\ -\ell_e \cos\alpha_e - d_e \sin\alpha_e \end{bmatrix}, \qquad \boldsymbol{f}_{se} = \begin{bmatrix} 0 \\ s_e \sin\beta_e \\ -s_e \cos\beta_e \end{bmatrix} \tag{23}$$

Expressions for the lift, drag, and side forces for all aircraft components, and their dependence on velocities at typical points, are given in Ref. 24. All aerodynamic forces are in terms of respective component body axes.

IV. Perturbation Approach to the Aircraft Dynamics and Control

Next, we turn our attention to the aircraft control problem. Controls are of two general types, one designed to fly the aircraft on a given flight path, defined by three translations and three rotations, and the other to suppress any deviations from the chosen path. The first type involves rigid-body motions of the aircraft, which are in general large, and the second type involves elastic deformations and perturbations in the rigid-body motions of the aircraft, which tend to be small compared to the rigid-body motions. Moreover, we observe that the state equations, Eqs. (14), are in general nonlinear and of high order, where the nonlinearity can be traced to the large rigid-body variables. On the other hand, the high order can be traced to the large number of small elastic variables. In view of this, a solution by a perturbation approach seems indicated, which amounts to a separation of the problem into a zero-order problem for the large variables and a first-order problem for the small variables, where the difference between the large and small variables is one order of magnitude or more. Physically, in the zero-order problem, the aircraft executes a given maneuver as if it were rigid, in which case the mathematical formulation consists of a maximum of six coupled, generally nonlinear second-order ordinary differential equations, three for rigid-body translations and three for rigid-body rotations. They correspond to the equations commonly used in flight dynamics. On the other hand, the first-order problem involves the elastic deformations, as well as small perturbations in the rigid-body variables. We refer to the first-order problem as an *extended perturbation problem* for reasons explained later in this section. We express the perturbation solution as follows:

$$\begin{gathered}
\boldsymbol{R}_f = \boldsymbol{R}_f^{(0)} + \boldsymbol{R}_f^{(1)}, \qquad \boldsymbol{\theta}_f = \boldsymbol{\theta}_f^{(0)} + \boldsymbol{\theta}_f^{(1)}, \qquad \boldsymbol{V}_f = \boldsymbol{V}_f^{(0)} + \boldsymbol{V}_f^{(1)} \\
\boldsymbol{\omega}_f = \boldsymbol{\omega}_f^{(0)} + \boldsymbol{\omega}_f^{(1)}, \qquad \boldsymbol{p}_{Vf} = \boldsymbol{p}_{Vf}^{(0)} + \boldsymbol{p}_{Vf}^{(1)}, \qquad \boldsymbol{p}_{\omega f} = \boldsymbol{p}_{\omega f}^{(0)} + \boldsymbol{p}_{\omega f}^{(1)} \\
\boldsymbol{F} = \boldsymbol{F}^{(0)} + \boldsymbol{F}^{(1)}, \qquad \boldsymbol{M} = \boldsymbol{M}^{(0)} + \boldsymbol{M}^{(1)}
\end{gathered} \tag{24}$$

where the superscripts (0) and (1) denote orders of magnitude. All the quantities related to the elastic deformations are regarded as being of first order. Then, inserting Eqs. (24) into the state equations, Eqs. (14), and separating different orders of magnitude, we obtain the zero-order problem, or the quasi-rigid flight dynamics problem

$$\begin{gathered}
\dot{\boldsymbol{R}}_f^{(0)} = C_f^{(0)T} \boldsymbol{V}_f^{(0)}, \qquad \dot{\boldsymbol{\theta}}_f^{(0)} = \left(E_f^{(0)}\right)^{-1} \boldsymbol{\omega}_f^{(0)} \\
\dot{\boldsymbol{p}}_{Vf}^{(0)} = -\tilde{\omega}_f^{(0)} \boldsymbol{p}_{Vf}^{(0)} + \boldsymbol{F}^{(0)}, \qquad \dot{\boldsymbol{p}}_{\omega f}^{(0)} = -\tilde{V}_f^{(0)} \boldsymbol{p}_{Vf}^{(0)} - \tilde{\omega}_f^{(0)} \boldsymbol{p}_{\omega f}^{(0)} + \boldsymbol{M}^{(0)}
\end{gathered} \tag{25}$$

in which $C_f^{(0)}$ and $E_f^{(0)}$ are as shown in Ref. 24. Moreover, from Eqs. (20), the zero-order generalized force and moment are given by

$$
\begin{aligned}
\boldsymbol{F}^{(0)} &= \int_{D_f} \left[\boldsymbol{f}_f^{(0)} + \boldsymbol{F}_E^{(0)} \delta(\boldsymbol{r} - \boldsymbol{r}_E) \right] \mathrm{d}D_f \\
&\quad + C_w^T \int_{D_w} \boldsymbol{f}_w^{(0)} \, \mathrm{d}D_w + C_e^T \int_{D_e} \boldsymbol{f}_e^{(0)} \, \mathrm{d}D_e \\
\boldsymbol{M}^{(0)} &= \int_{D_f} \tilde{r}_f \left[\boldsymbol{f}_f^{(0)} + \boldsymbol{F}_E^{(0)} \delta(\boldsymbol{r} - \boldsymbol{r}_E) \right] \mathrm{d}D_f \\
&\quad + \int_{D_w} \left(\tilde{r}_{fw} C_w^T + C_w^T \tilde{r}_w \right) \boldsymbol{f}_w^{(0)} \, \mathrm{d}D_w \\
&\quad + \int_{D_e} \left(\tilde{r}_{fe} C_e^T + C_e^T \tilde{r}_e \right) \boldsymbol{f}_e^{(0)} \, \mathrm{d}D_e
\end{aligned}
\tag{26}
$$

where the zero-order parts of the aerodynamic force densities contributing to $\boldsymbol{f}_f^{(0)}$, $\boldsymbol{f}_w^{(0)}$, and $\boldsymbol{f}_e^{(0)}$ can be found in Ref. 24. The zero-order state is defined as $\boldsymbol{x}^{(0)} = [\boldsymbol{R}_f^{(0)T} \ \boldsymbol{\theta}_f^{(0)T} \ \boldsymbol{p}_{Vf}^{(0)T} \ \boldsymbol{p}_{\omega f}^{(0)T}]^T$, and Eqs. (25) contain in addition $\boldsymbol{V}_f^{(0)}$ and $\boldsymbol{\omega}_f^{(0)}$, which, from Eqs. (13), are related to $\boldsymbol{p}_{Vf}^{(0)}$ and $\boldsymbol{p}_{\omega f}^{(0)}$ by

$$
\boldsymbol{p}_{Vf}^{(0)} = m \boldsymbol{V}_f^{(0)} + \tilde{S}^{(0)T} \boldsymbol{\omega}_f^{(0)}, \qquad \boldsymbol{p}_{\omega f}^{(0)} = \tilde{S}^{(0)} \boldsymbol{V}_f^{(0)} + J^{(0)} \boldsymbol{\omega}_f^{(0)} \tag{27}
$$

In a direct solution, the forces $\boldsymbol{f}_i^{(0)}$, $i = f, w, e$, and $\boldsymbol{F}_E^{(0)}$ are given and Eqs. (25–27) are solved for the state $\boldsymbol{x}^{(0)}$, but this is not how a pilot flies an aircraft. Indeed, the pilot begins with a flight plan and sets the controls, i.e., the engine throttle and the control surfaces angles, to realize the preselected flight plan. In dynamic terms, this amounts to first selecting the state $\boldsymbol{x}^{(0)}$ and then solving for the control contributions to $\boldsymbol{f}_i^{(0)}$, $i = f, w, e$, and $\boldsymbol{F}_E^{(0)}$. This describes an inverse dynamics problem, which is much simpler than the direct problem because it requires the solution of nonlinear algebraic equations instead of nonlinear differential equations.

Inserting Eqs. (24) into Eqs. (14) and retaining the first-order terms, we obtain the first-order problem defined by

$$
\begin{aligned}
\dot{\boldsymbol{R}}_f^{(1)} &= C_f^{(0)T} \boldsymbol{V}_f^{(1)} + C_f^{(1)T} \boldsymbol{V}_f^{(0)} \\
\dot{\boldsymbol{\theta}}_f^{(1)} &= \left(E_f^{(0)} \right)^{-1} \boldsymbol{\omega}_f^{(1)} - \left(E_f^{(0)} \right)^{-1} E_f^{(1)} \left(E_f^{(0)} \right)^{-1} \boldsymbol{\omega}_f^{(0)} \\
\dot{\boldsymbol{q}}_{ui} &= \boldsymbol{s}_{ui}, \qquad \dot{\boldsymbol{q}}_{\psi i} = \boldsymbol{s}_{\psi i}, \qquad i = f, w, e \\
\dot{\boldsymbol{p}}_{Vf}^{(1)} &= -\tilde{\omega}_f^{(1)} \boldsymbol{p}_{Vf}^{(0)} - \tilde{\omega}_f^{(0)} \boldsymbol{p}_{Vf}^{(1)} + \boldsymbol{F}^{(1)} \\
\dot{\boldsymbol{p}}_{\omega f}^{(1)} &= -\tilde{V}_f^{(1)} \boldsymbol{p}_{Vf}^{(0)} - \tilde{V}_f^{(0)} \boldsymbol{p}_{Vf}^{(1)} - \tilde{\omega}_f^{(1)} \boldsymbol{p}_{\omega f}^{(0)} - \tilde{\omega}_f^{(0)} \boldsymbol{p}_{\omega f}^{(1)} + \boldsymbol{M}^{(1)} \\
\dot{\boldsymbol{p}}_{uf} &= \int \Phi_{uf}^T \left(\tilde{\omega}_f^{(0)T} \bar{\boldsymbol{V}}_f^{(1)} + \tilde{\omega}_f^{(1)T} \bar{\boldsymbol{V}}_f^{(0)} \right) \mathrm{d}m_f \\
&\quad + \int \Phi_{ufw}^T \left(\tilde{\omega}_f^{(0)T} C_w^T \bar{\boldsymbol{V}}_w^{(1)} + \tilde{\omega}_f^{(1)T} C_w^T \bar{\boldsymbol{V}}_w^{(0)} \right) \mathrm{d}m_w \\
&\quad + \int \Phi_{ufe}^T \left(\tilde{\omega}_f^{(0)T} C_e^T \bar{\boldsymbol{V}}_e^{(1)} + \tilde{\omega}_f^{(1)T} C_e^T \bar{\boldsymbol{V}}_e^{(0)} \right) \mathrm{d}m_e \\
&\quad - K_{uf} \boldsymbol{q}_{uf} - C_{uf} \boldsymbol{s}_{uf} + \boldsymbol{Q}_{uf} + \int \Phi_{uf}^T \tilde{\omega}_f^{(0)T} \bar{\boldsymbol{V}}_f^{(0)} \, \mathrm{d}m_f \\
&\quad + \int \Phi_{ufw}^T \tilde{\omega}_f^{(0)T} C_w^T \bar{\boldsymbol{V}}_w^{(0)} \, \mathrm{d}m_w + \int \Phi_{ufe}^T \tilde{\omega}_f^{(0)T} C_e^T \bar{\boldsymbol{V}}_e^{(0)} \, \mathrm{d}m_e \\
\dot{\boldsymbol{p}}_{\psi f} &= -K_{\psi f} \boldsymbol{q}_{\psi f} - C_{\psi f} \boldsymbol{s}_{\psi f} + \boldsymbol{Q}_{\psi f} \\
\dot{\boldsymbol{p}}_{ui} &= \int \Phi_{ui}^T \left(\widetilde{C_i \boldsymbol{\omega}_f^{(0)}}^T \bar{\boldsymbol{V}}_i^{(1)} + \widetilde{C_i \boldsymbol{\omega}_f^{(1)}}^T \bar{\boldsymbol{V}}_i^{(0)} \right) \mathrm{d}m_i - K_{ui} \boldsymbol{q}_{ui} \\
&\quad - C_{ui} \boldsymbol{s}_{ui} + \boldsymbol{Q}_{ui} + \int \Phi_{ui}^T \widetilde{C_i \boldsymbol{\omega}_f^{(0)}}^T \bar{\boldsymbol{V}}_i^{(0)} dm_i, \qquad i = w, e \\
\dot{\boldsymbol{p}}_{\psi i} &= -K_{\psi i} \boldsymbol{q}_{\psi i} - C_{\psi i} \boldsymbol{s}_{\psi i} + \boldsymbol{Q}_{\psi i}, \qquad i = w, e
\end{aligned}
\tag{28}
$$

where, from Eqs. (20), the first-order generalized forces are

$$
\begin{aligned}
\boldsymbol{F}^{(1)} &= \int_{D_f} \left[\boldsymbol{f}_f^{(1)} + \boldsymbol{F}_E^{(1)} \delta(\boldsymbol{r} - \boldsymbol{r}_E) \right] \mathrm{d}D_f \\
&\quad + C_w^T \int_{D_w} \boldsymbol{f}_w^{(1)} \, \mathrm{d}D_w + C_e^T \int_{D_e} \boldsymbol{f}_e^{(1)} \, \mathrm{d}D_e \\
\boldsymbol{M}^{(1)} &= \int_{D_f} \left\{ \tilde{r}_f \left[\boldsymbol{f}_f^{(1)} + \boldsymbol{F}_E^{(1)} \delta(\boldsymbol{r} - \boldsymbol{r}_E) \right] \right. \\
&\quad \left. + \widetilde{\Phi_{uf} \boldsymbol{q}_{uf}} \left[\boldsymbol{f}_f^{(0)} + \boldsymbol{F}_E^{(0)} \delta(\boldsymbol{r} - \boldsymbol{r}_E) \right] \right\} \mathrm{d}D_f \\
&\quad + \int_{D_w} \left[\left(\tilde{r}_{fw} C_w^T + C_w^T \tilde{r}_w \right) \boldsymbol{f}_w^{(1)} + \left(\widetilde{\Phi_{ufw} \boldsymbol{q}_{uf}} C_w^T \right. \right. \\
&\quad \left. \left. + C_w^T \widetilde{\Phi_{uw} \boldsymbol{q}_{uw}} \right) \boldsymbol{f}_w^{(0)} \right] \mathrm{d}D_w + \int_{D_e} \left[\left(\tilde{r}_{fe} C_e^T + C_e^T \tilde{r}_e \right) \boldsymbol{f}_e^{(1)} \right. \\
&\quad \left. + \left(\widetilde{\Phi_{ufe} \boldsymbol{q}_{uf}} C_e^T + C_e^T \widetilde{\Phi_{ue} \boldsymbol{q}_{ue}} \right) \boldsymbol{f}_e^{(0)} \right] \mathrm{d}D_e \\
\boldsymbol{Q}_{uf} &= \int_{D_f} \Phi_{uf}^T \left[\boldsymbol{f}_f^{(0)} + \boldsymbol{f}_f^{(1)} + \boldsymbol{F}_E^{(0)} \delta(\boldsymbol{r} - \boldsymbol{r}_E) + \boldsymbol{F}_E^{(1)} \delta(\boldsymbol{r} - \boldsymbol{r}_E) \right] \mathrm{d}D_f \\
&\quad + \int_{D_w} \left(\tilde{r}_w^T C_w \Delta \Phi_{ufw} + C_w \Phi_{ufw} \right)^T \left(\boldsymbol{f}_w^{(0)} + \boldsymbol{f}_w^{(1)} \right) \mathrm{d}D_w \\
&\quad + \int_{D_e} \left(\tilde{r}_e^T C_e \Delta \Phi_{ufe} + C_e \Phi_{ufe} \right)^T \left(\boldsymbol{f}_e^{(0)} + \boldsymbol{f}_e^{(1)} \right) \mathrm{d}D_e \\
\boldsymbol{Q}_{\psi f} &= \int_{D_f} \Phi_{\psi f}^T \tilde{r}_f \left[\boldsymbol{f}_f^{(0)} + \boldsymbol{f}_f^{(1)} + \boldsymbol{F}_E^{(0)} \delta(\boldsymbol{r} - \boldsymbol{r}_E) + \boldsymbol{F}_E^{(1)} \delta(\boldsymbol{r} - \boldsymbol{r}_E) \right] \mathrm{d}D_f \\
&\quad + \int_{D_w} \left(\tilde{r}_w^T C_w \Phi_{\psi fw} + C_w \tilde{r}_{fw}^T \Phi_{\psi fw} \right)^T \left(\boldsymbol{f}_w^{(0)} + \boldsymbol{f}_w^{(1)} \right) \mathrm{d}D_w \\
&\quad + \int_{D_e} \left(\tilde{r}_e^T C_e \Phi_{\psi fe} + C_e \tilde{r}_{fe}^T \Phi_{\psi fe} \right)^T \left(\boldsymbol{f}_e^{(0)} + \boldsymbol{f}_e^{(1)} \right) \mathrm{d}D_e \\
\boldsymbol{Q}_{ui} &= \int_{D_i} \Phi_{ui}^T \left(\boldsymbol{f}_i^{(0)} + \boldsymbol{f}_i^{(1)} \right) \mathrm{d}D_i, \qquad i = w, e \\
\boldsymbol{Q}_{\psi i} &= \int_{D_i} \Phi_{\psi i}^T \tilde{r}_i \left(\boldsymbol{f}_i^{(0)} + \boldsymbol{f}_i^{(1)} \right) \mathrm{d}D_i, \qquad i = w, e
\end{aligned}
\tag{29}
$$

Equations (28) and (29) must be solved for the state $\boldsymbol{x}^{(1)} = [\boldsymbol{R}_f^{(1)T} \ \boldsymbol{\theta}_f^{(1)T} \ \boldsymbol{q}_{uf}^T \ \boldsymbol{q}_{uw}^T \ \cdots \ \boldsymbol{q}_{\psi e}^T \ \boldsymbol{p}_{Vf}^{(1)T} \ \boldsymbol{p}_{\omega f}^{(1)T} \ \boldsymbol{p}_{uf}^T \ \boldsymbol{p}_{uw}^T \ \cdots \ \boldsymbol{p}_{\psi e}^T]^T$ in conjunction with, from Eq. (12), the first-order momentum

$$
\boldsymbol{p}^{(1)} = M^{(0)} \boldsymbol{V}^{(1)} + M^{(1)} \boldsymbol{V}^{(0)} \tag{30}
$$

in which $M^{(0)}$ is the mass matrix of the quasi-rigid aircraft and $M^{(1)}$ is the part of the mass matrix due to the elastic deformations. We note that, for brevity, expressions for many of the quantities entering into Eqs. (28–30) were not given; they can be found in Ref. 24.

Because we referred to the zero-order problem, Eqs. (25–27), as the flight dynamics problem, it is tempting to think of the first-order problem, Eqs. (28–30), as the aeroservoelasticity problem. A closer examination of Eqs. (28–30), however, reveals that the first-order problem is considerably broader in scope than the commonly encountered aeroservoelasticity. Indeed, Eqs. (28–30) are characterized by the following features: 1) They are concerned with whole flexible aircraft, and not with wings alone. 2) All six rigid-body degrees of freedom are automatically included. 3) The motions are expressed in terms of components along noninertial body axes to accommodate all possible aircraft maneuvers, as opposed to wings fixed at the root, or wings with the root translating uniformly, which are dynamically equivalent and require the use of inertial reference frames only. 4) They contain zero-order terms, so that they receive inputs from aircraft maneuvers, which further implies that there is a different set of extended perturbation equations for every conceivable maneuver, not just a single set. 5) The task of ensuring that the aircraft flies smoothly along the intended path is carried out by feedback control in conjunction with a stochastic state estimator. (See Sec. V.) Note that feedback controls prevent divergence and flutter automatically.

We observe that the first-order equations representing the extended aeroelasticity problem are linear and tend to be of high order. Moreover, they contain the zero-order variables $\boldsymbol{V}_f^{(0)}$ and $\boldsymbol{\omega}_f^{(0)}$, representing a given maneuver, as coefficients and as an input. If $\boldsymbol{V}_f^{(0)}$ and $\boldsymbol{\omega}_f^{(0)}$ are constant, then the system is time-invariant, and if $\boldsymbol{V}_f^{(0)}$ and $\boldsymbol{\omega}_f^{(0)}$ depend on time, then the system is time-varying. In either case, controls can be designed by various methods. In the time-invariant case, a stability analysis for the closed-loop system can be carried out by solving an eigenvalue problem. Such a stability analysis is precluded in the time-varying case. Simulation of the response of the closed-loop system to external excitations, such as gusts, can be obtained in both the time-invariant and time-varying cases.

V. Control Design

Flying aircraft are subjected to various disturbances tending to drive them from the intended flight path and to cause vibration. If the system is controllable,[26] these effects can be suppressed through controls that are carried out by means of actuators; in the case of aircraft, they consist of the engines throttles and the control surfaces. In practice, controllability can be ascertained on physical grounds by making sure that the input forces, namely, the forces due to the engine thrust and control surfaces, affect all the state variables.

As indicated in Sec. IV, there are two types of controls, one type designed to permit the aircraft to fly on a desired flight path as if it were rigid and the other type to reduce any deviations from the rigid-body flight path to zero, which amounts to suppressing vibration and perturbations in the rigid-body motions of the aircraft. The first is associated with the flight dynamics problem and the second with the extended aeroelasticity problem. In general, the engines throttles and control surfaces angles are set so as to ensure that the aircraft is able to carry out the required maneuvers, as well as to suppress any undesirable disturbances, thus addressing the needs of both the flight dynamics problem and extended perturbation problem.

Using Eqs. (25), we write the flight dynamics problem, or the zero-order problem, in the compact state form

$$\dot{\boldsymbol{x}}^{(0)}(t) = \boldsymbol{f}\left[\boldsymbol{x}^{(0)}(t), \boldsymbol{V}_{rb}^{(0)}(t)\right] + B^{(0)}\left[\boldsymbol{V}_{rb}^{(0)}(t)\right]\boldsymbol{u}^{(0)}(t) \tag{31}$$

which must be considered in conjunction with Eqs. (27), where $\boldsymbol{f}$ is a nonlinear function, in which $\boldsymbol{V}_{rb}^{(0)} = [\boldsymbol{V}_f^{(0)^T} \quad \boldsymbol{\omega}_f^{(0)^T}]^T$, $B^{(0)}$ is a coefficient matrix, and $\boldsymbol{u}^{(0)} = [F_E^{(0)} \quad \delta_a^{(0)} \quad \delta_e^{(0)} \quad \delta_r^{(0)}]^T$ is the control vector. In the context of the present integrated approach, Eqs. (27) and (31) represent an inverse dynamics problem, in the sense that a state vector $\boldsymbol{x}^{(0)}$ describing a desired maneuver is postulated and a force vector $\boldsymbol{u}^{(0)}$ permitting realization of the given maneuver is determined.

Next, we assume that $\boldsymbol{x}^{(0)}(t)$ and $\boldsymbol{V}_{rb}^{(0)}(t)$ are known and use Eqs. (28) to express the extended perturbation problem, or first-order problem, in the form

$$\dot{\boldsymbol{x}}^{(1)}(t) = A(t)\boldsymbol{x}^{(1)}(t) + B(t)\boldsymbol{u}^{(1)}(t) + [0 \quad I]^T \boldsymbol{F}_{\text{ext}}(t) \tag{32}$$

in which $A(t) = A[\boldsymbol{x}^{(0)}(t), \boldsymbol{V}_{rb}^{(0)}(t)]$ and $B(t) = B\ [\boldsymbol{x}^{(0)}(t), \boldsymbol{V}_{rb}^{(0)}(t)]$ are coefficient matrices, $\boldsymbol{u}^{(1)}(t) = [F_E^{(1)} \quad \delta_a^{(1)} \quad \delta_e^{(1)} \quad \delta_r^{(1)}]^T$ is a first-order control vector, and $\boldsymbol{F}_{\text{ext}}$ is an external disturbing force vector of a transient nature. Equations (32) must be solved in conjunction with Eq. (30).

Equation (32) represents a set of linear equations, and the objective is to find a control vector $\boldsymbol{u}^{(1)}(t)$ that drives the state vector $\boldsymbol{x}^{(1)}$ to zero. To this end, we consider a linear quadratic regulator (LQR) in which the objective is to determine an optimal control vector minimizing a quadratic performance measure consisting of two parts, one representing a measure of the distance of the state from the origin of the state space and the other a measure of the control effort.[26] It is shown in Ref. 26 that the optimal feedback control vector is given by

$$\boldsymbol{u}^{(1)}(t) = -G(t)\boldsymbol{x}^{(1)}(t) \tag{33}$$

where $G(t)$ is the control gain matrix; it can be obtained by solving a transient matrix Riccati equation, a nonlinear matrix differential equation to be integrated backward in time.[26] Inserting Eq. (33) into Eq. (32), we obtain the closed-loop equation

$$\dot{\boldsymbol{x}}^{(1)}(t) = [A(t) - B(t)G(t)]\boldsymbol{x}^{(1)}(t) + [0 \quad I]^T \boldsymbol{F}_{\text{ext}}(t) \tag{34}$$

which can be integrated to simulate the system response.

If the zero-order solution is constant, the matrix A is constant. Then, under certain circumstances,[26] the gain matrix G becomes constant, so that the closed-loop equation reduces to one with constant coefficients, or

$$\dot{\boldsymbol{x}}^{(1)}(t) = (A - BG)\boldsymbol{x}^{(1)}(t) + [0 \quad I]^T \boldsymbol{F}_{\text{ext}}(t) \tag{35}$$

which can be used for response simulations. For a stability analysis, we solve the associated eigenvalue problem for the matrix $A - BG - \lambda I$. The closed-loop system is asymptotically stable if all the eigenvalues are real and negative and/or complex with negative real part.

The control vector $\boldsymbol{u}^{(1)}$ is optimal in the sense that it minimizes the performance measure, but the physical merit of this optimality is debatable. In fact, it is often necessary to adjust the weighting matrices in the quadratic performance measure to achieve a desirable system performance. The real value of the LQR algorithm is that it guarantees a stable closed-loop system.

Implementation of the control law, Eq. (33), requires knowledge of the state vector $\boldsymbol{x}^{(1)}$, which can be obtained through measurements. This creates somewhat of a problem because measurements represent real quantities and our state vector consists of abstract generalized coordinates rather than real coordinates. Moreover, we feed back only perturbations from the maneuver variables. In the absence of external forces, the state equations describing the extended perturbation problem have the vector form

$$\dot{\boldsymbol{x}}^{(1)}(t) = A\boldsymbol{x}^{(1)}(t) + B\boldsymbol{u}^{(1)}(t) \tag{36}$$

Then, denoting the measurement vector by $\boldsymbol{y}(t)$, we can write

$$\boldsymbol{y}(t) = \boldsymbol{y}^{(0)}(t) + \boldsymbol{y}^{(1)}(t) \tag{37}$$

where $\boldsymbol{y}^{(0)}(t)$ is the contribution from the maneuver variables and $\boldsymbol{y}^{(1)}(t)$ is the contribution from the perturbations in the maneuver variables and the elastic variables. We express the latter in the form

$$\boldsymbol{y}^{(1)}(t) = C\boldsymbol{x}^{(1)}(t) \tag{38}$$

and refer to $\boldsymbol{y}^{(1)}(t)$ as the output vector. The assumption is made here that the system is observable.[26] In practice, the choice of sensors ensuring observability must be made on physical grounds, and it must be such that the sensors signals permit reconstruction of the state at all times.

In reality, the state vector cannot be determined exactly from the output vector and must be estimated. A device permitting an estimate of the state vector is known as an *observer*, or an *estimator*,[26] and can be expressed in the form

$$\dot{\hat{\boldsymbol{x}}}^{(1)}(t) = A\hat{\boldsymbol{x}}^{(1)}(t) + B\boldsymbol{u}^{(1)}(t) + K_o[\boldsymbol{y}^{(1)}(t) - C\hat{\boldsymbol{x}}^{(1)}(t)] \tag{39}$$

where K_o is an observer gain matrix. Subtracting Eq. (39) from Eq. (36), we obtain

$$\dot{\boldsymbol{e}}(t) = [A - K_o C]\boldsymbol{e}(t) \tag{40}$$

in which $\boldsymbol{e}(t) = \boldsymbol{x}^{(1)}(t) - \hat{\boldsymbol{x}}^{(1)}(t)$ represents the observer error vector. The objective is to find a matrix K_o such that the vector $\boldsymbol{e}(t)$ approaches zero as t increases. In this case, $\hat{\boldsymbol{x}}(t) \rightarrow \boldsymbol{x}(t)$ with time. For a time-invariant system, this amounts to determining K_o so that all of the eigenvalues of the matrix $A - K_o C$, known as the observer poles, lie in the left half of the complex plane. In implementing feedback controls, we must use the estimated state $\hat{\boldsymbol{x}}^{(1)}(t)$ because the actual state $\boldsymbol{x}^{(1)}(t)$ is not available. Hence, the control law, Eq. (33), must be replaced by

$$\boldsymbol{u}^{(1)}(t) = -G\hat{\boldsymbol{x}}^{(1)}(t) \tag{41}$$

An optimal observer gain matrix can be obtained by the adoption of a stochastic approach, leading to the so-called *Kalman–Bucy filter* (see Ref. 26). The optimal gain matrix K_o is determined by minimizing a quadratic performance measure in terms of the error vector $\boldsymbol{e}(t)$. This, in turn, requires the solution of another Riccati equation involving two noise intensity matrices, one associated with the actuators noise and the other with the sensors noise.[26]

To determine the observer state $\hat{\boldsymbol{x}}^{(1)}$, we must integrate Eq. (39), which requires the output matrix C. In the case at hand, the determination of C is very complex and tedious, so that we do not pursue it here. Complete details can be found in Ref. 24.

VI. Numerical Example

The flight of flexible aircraft is fully described by Eqs. (25–30). A solution of these equations requires the aircraft geometry, the mass and stiffness distributions, and the aerodynamic coefficients. Information pertaining to an actual aircraft was made available by an aircraft manufacturer and is provided in Ref. 24.

The solution of the flight dynamics equations, Eqs. (25–27), requires numerical values for various quantities, including the aerodynamic forces. Some of them are as follows: engine locations, $\boldsymbol{r}_{E1} = [-9.05 \;\; 3.08 \;\; -1.16]^T$ ft, $\boldsymbol{r}_{E2} = [-9.05 \;\; -3.08 \;\; -1.16]^T$ ft; total aircraft mass, $m = 403.0752 \text{ lb} \cdot \text{s}^2/\text{ft}$. The remaining quantities are given in Ref. 24. The flight dynamics problem essentially consists of the adjustment of the control surfaces angles and the engine throttle for a given aircraft maneuver. It amounts to the problem of "trimming" the aircraft. As far as this paper is concerned, it provides the input $\boldsymbol{V}_f^{(0)}$, $\boldsymbol{\omega}_f^{(0)}$, $\boldsymbol{p}_{Vf}^{(0)}$, and $\boldsymbol{p}_{\omega f}^{(0)}$ to the extended aeroelasticity problem.

The solution of the extended aeroelasticity equations, Eqs. (28–30), requires an explicit choice of the structural model for the aircraft of Fig. 1. As a first approximation, the fuselage, wing, and both the horizontal and vertical stabilizers in the empennage are modeled as beams clamped at the origin of the respective body axes and undergoing bending and torsion. The fuselage undergoes the bending displacements u_{fy} and u_{fz} and the torsional displacement ψ_{fx}, as shown in Fig. 2, so that $\boldsymbol{u}_f = [0 \;\; u_{fy} \;\; u_{fz}]^T$ and $\boldsymbol{\psi}_f = [\psi_{fx} \;\; 0 \;\; 0]^T$.

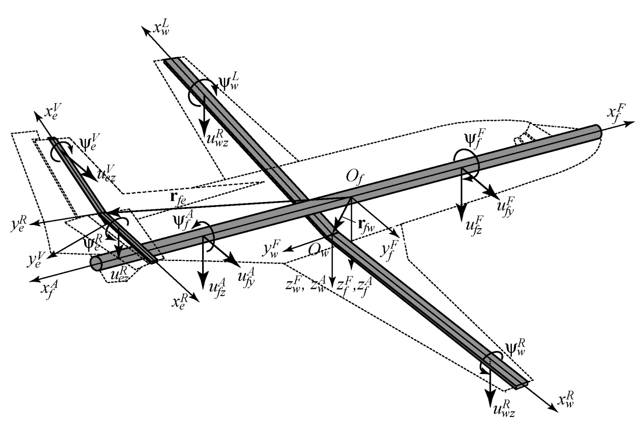

Fig. 2 Aircraft components undergoing bending and torsion.

On the other hand, the wing and the stabilizers undergo only one bending and one torsional displacement each. Note that, as customary, sending displacements are measured relative to the elastic axis. Each clamped beam is assumed to be discretized by the Galerkin method in conjunction with two shape functions per displacement component. For bending, the shape functions are chosen as the eigenfunctions of a uniform cantilever beam

$$\phi_{uir} = \sin\beta_r x_i - \sinh\beta_r x_i - \frac{\sin\beta_r L_i + \sinh\beta_r L_i}{\cos\beta_r L_i + \cosh\beta_r L_i} \times (\cos\beta_r x_i - \cosh\beta_r x_i), \qquad r = 1, 2, \quad i = f, w, e \tag{42}$$

and for torsion, the eigenfunctions of a uniform clamped-free shaft

$$\phi_{\psi ir} = \sin(2r - 1)\pi x_i / 2L_i, \qquad r = 1, 2, \qquad i = f, w, e \tag{43}$$

where L_i is the length of the cantilever beam. In this regard, we note from Fig. 2 that the fuselage is modeled as two cantilever beams clamped at O_f, one pointing to the aircraft nose and the other to the tail. New quantities entering into the first-order equations are $\boldsymbol{F}^{(1)}$, $\boldsymbol{M}^{(1)}$, $\boldsymbol{Q}_{ui}$, and $\boldsymbol{Q}_{\psi i}$, $i = f, w, e$, which are given by Eqs. (29), and Φ_{ui} and $\Phi_{\psi i}$, which contain ϕ_{uir} and $\phi_{\psi ir}$ as given in Eqs. (42) and (43), respectively. Moreover, the stiffness matrices are obtained from the potential energy, as follows:

$$\begin{aligned} V = {} & \frac{1}{2}\int_{D_f}\left[EI_{fz}\left(\frac{\partial^2 u_{fy}}{\partial x_f^2}\right)^2 + EI_{fy}\left(\frac{\partial^2 u_{fz}}{\partial x_f^2}\right)^2\right. \\ & \left. + GJ_{fx}\left(\frac{\partial \psi_{fx}}{\partial x_f}\right)^2\right]\mathrm{d}D_f + \int_{D_w}\left[EI_w\left(\frac{\partial^2 u_{wz}}{\partial x_w^2}\right)^2\right. \\ & \left. + GJ_w\left(\frac{\partial \psi_{wx}}{\partial x_w}\right)^2\right]\mathrm{d}D_w + \int_{D_e}\left[EI_e\left(\frac{\partial^2 u_{ez}}{\partial x_e^2}\right)^2\right. \\ & \left. + GJ_e\left(\frac{\partial \psi_{ex}}{\partial x_e}\right)^2\right]\mathrm{d}D_e = \frac{1}{2}\sum_i\left(\boldsymbol{q}_{ui}^T K_{ui}\boldsymbol{q}_{ui} + \boldsymbol{q}_{\psi i}^T K_{\psi i}\boldsymbol{q}_{\psi i}\right) \end{aligned} \tag{44}$$

where

$$\begin{aligned} K_{uf} &= \int_{D_f}(\Phi_{uf}'')^T \operatorname{diag}[EI_{fz} \quad EI_{fy}]\Phi_{uf}'' \,\mathrm{d}D_f \\ K_{ui} &= \int_{D_i} EI_i(\Phi_{ui}'')^T\Phi_{ui}''\,\mathrm{d}D_i, \qquad i = f, w, e \\ K_{\psi i} &= \int_{D_i} GJ_i(\Phi_{\psi i}')^T\Phi_{\psi i}'\,\mathrm{d}D_i, \qquad i = f, w, e \end{aligned} \tag{45}$$

are the desired stiffness matrices, in which primes denote differentiations with respect to x_i.

Equation (32) for the model in question is of order 76, but the vector equation is linear. However, in the case of certain aircraft maneuvers, the system is time-varying.

We consider two cases, steady level flight and a level steady turn maneuver.

A. Steady Level Flight

For steady level flight, the zero-order velocities are defined by

$$\boldsymbol{V}_f^{(0)} = C_f^{(0)}[V^{(0)} \quad 0 \quad 0]^T = \text{constant}, \qquad \boldsymbol{\omega}_f^{(0)} = \boldsymbol{0} \tag{46}$$

where $V^{(0)}$ is the aircraft forward velocity. From Eqs. (27), we conclude that the zero-order momenta are

$$\boldsymbol{p}_{Vf}^{(0)} = m\boldsymbol{V}_f^{(0)} = \text{constant}, \qquad \boldsymbol{p}_{\omega f}^{(0)} = \tilde{S}^{(0)}\boldsymbol{V}_f^{(0)} = \text{constant} \tag{47}$$

Hence, from the second line of Eqs. (25), we have

$$\boldsymbol{F}^{(0)} = \boldsymbol{0}, \qquad \boldsymbol{M}^{(0)} = \boldsymbol{0} \tag{48}$$

The implication of Eqs. (48) is that, for steady level flight, the forces and moments due to the engine thrust, aerodynamic forces, gravitational forces, and control forces balance out to zero. The angle of attack can be expressed as $\alpha_f^{(0)} = \tan^{-1}(V_{fz}^{(0)}/V_{fx}^{(0)}) = \alpha_0 =$ constant. For level flight, we have $\psi^{(0)} = \phi^{(0)} = 0$, so that the pitch angle is equal to the angle of attack, or $\theta^{(0)} = \alpha_f^{(0)}$. Moreover, because $V_{fy}^{(0)} = 0$, the sideslip angle is zero, $\beta_f^{(0)} = \tan^{-1}(V_{fy}^{(0)}/V_{fx}^{(0)}) = 0$. In view of this, and due to the symmetry of the gravitational and aerodynamic forces, the side force $F_y^{(0)}$ and the roll and yaw moments, $M_x^{(0)}$ and $M_z^{(0)}$, are automatically zero. We assume that $V^{(0)} = 416.67$ in./s and consider a flight at a 25,000-ft altitude, so that the speed of sound is 1016.1 ft/s and, hence, the Mach number is $416.67/1016.1 = 0.41$. From the first of Eqs. (46), we have $\boldsymbol{V}_f^{(0)} = 416.67[\cos\theta^{(0)} \;\; 0 \;\; \sin\theta^{(0)}]^T$. Then, using Eqs. (26), Eqs. (48) yield a set of nonlinear algebraic equations for $F_x^{(0)}$, $F_z^{(0)}$, and $M_y^{(0)}$, which can be solved for the pitch angle $\theta^{(0)}$, the engine thrust $F_E^{(0)}$, and the elevator angle $\delta_e^{(0)}$. Solving the nonlinear equations, we obtain $\theta^{(0)} = 3.8216$ deg, $F_E^{(0)} = 431.6465$ lb, and $\delta_e^{(0)} = -15.4870$ deg, so that the zero-order control vector is given by $\boldsymbol{u}^{(0)} = [F_E^{(0)} \;\; 0 \;\; \delta_e^{(0)} \;\; 0]^T = [431.6465 \;\; 0 \;\; -15.4870 \text{ deg} \;\; 0]^T$.

In the case of steady level flight, the aircraft experiences small static deformations due to zero-order forces.[24] Assuming that the first-order state $\boldsymbol{x}^{(1)}$ is measured from the static elastic displacement position, the first-order state equations for steady level flight can be written in the customary form, Eq. (32), where A and B are constant coefficient matrices. This requires the mass, damping, and stiffness matrices.

Next, we consider feedback control design by the LQR and LQG methods, in sequence. Assuming that the damping functions c_{ui} and $c_{\psi i}$, $i = f, w, e$, are all constant, the damping matrices, Eqs. (11), reduce to $C_{ui} = c_{ui}K_{ui}$, $C_{\psi i} = c_{\psi i}K_{\psi i}$, $i = f, w, e$, which state that the damping matrices are proportional to the stiffness matrices. Numerical values of the mass, damping, and stiffness matrices are given in Ref. 24. Then, computing the coefficient matrices A and B, assuming that in the feedback control vector $\boldsymbol{u}^{(1)} = [F_E^{(1)} \;\; \delta_a^{(1)} \;\; \delta_e^{(1)} \;\; \delta_r^{(1)}]^T$ the right and the left ailerons rotate by angles of the same magnitude $\delta_a^{(1)}$, but of opposite sense, and that the right and left elevators rotate by angles of the same magnitude and sense, choosing the weighting matrices in the performance index, and solving the steady-state Riccati equation, we obtain the gain matrix G. Then, solving the closed-loop eigenvalue problem, we obtain the closed-loop eigenvalues

$$\lambda_{1,2} = -0.1038 \pm 0.0868i, \qquad \lambda_{3,4} = -0.1174 \pm 0.2033i$$
$$\lambda_5 = -0.2349, \qquad \lambda_{6,7} = -0.2918 \pm 0.3064i,$$
$$\ldots, \lambda_{75,76} = -211.4842 \pm 1063.7436i \tag{49}$$

Clearly, all the eigenvalues are real and negative or complex with negative real part, so that the closed-loop first-order system is asymptotically stable. The implication is that any disturbances from the steady level flight are driven to zero. This is in contrast with the open-loop eigenvalues, the eigenvalues of A, the first four of which are zero and the fifth of which is real and positive.

Finally, we consider the response of a closed-loop system to a gust acting on the wing and having the linearly distributed form

$$\boldsymbol{f}_w^R(x_w^R, t) = \begin{bmatrix} 0 & 0 & -0.5\left(3 + x_w^R/L_w\right)\delta(t) \end{bmatrix}^T, \quad 0 < x_w^R < L_w$$
$$\boldsymbol{f}_w^L(x_w^L, t) = \begin{bmatrix} 0 & 0 & -0.5\left(3 - x_w^L/L_w\right)\delta(t) \end{bmatrix}^T, \quad 0 < x_w^L < L_w \tag{50}$$

Inserting Eqs. (50) into Eqs. (29), we obtain the generalized force components of the disturbance vector $\boldsymbol{F}_{\text{ext}}$ entering into Eq. (35), which is integrated to obtain the system response. Figures 3 and 4 show plots of the response for a selected number of rigid-body and elastic variables, as well as of the control inputs. Plots for the response without and with the use of an observer are labeled actual and estimated, respectively. We note that convergence of the estimated state and control variables to the actual ones is relatively fast. Details of the observer design can be found in Ref. 24.

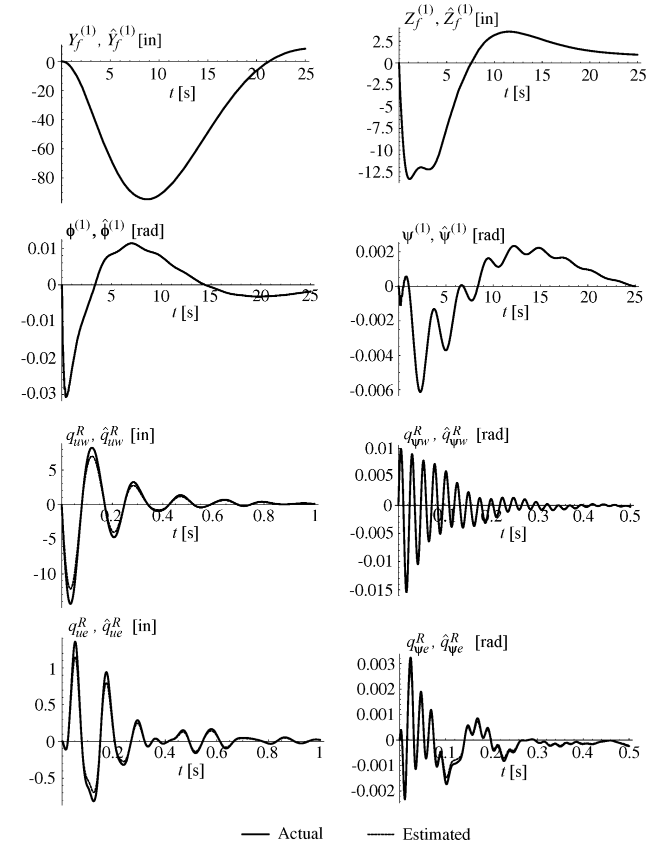

Fig. 3 Rigid-body and generalized displacements.

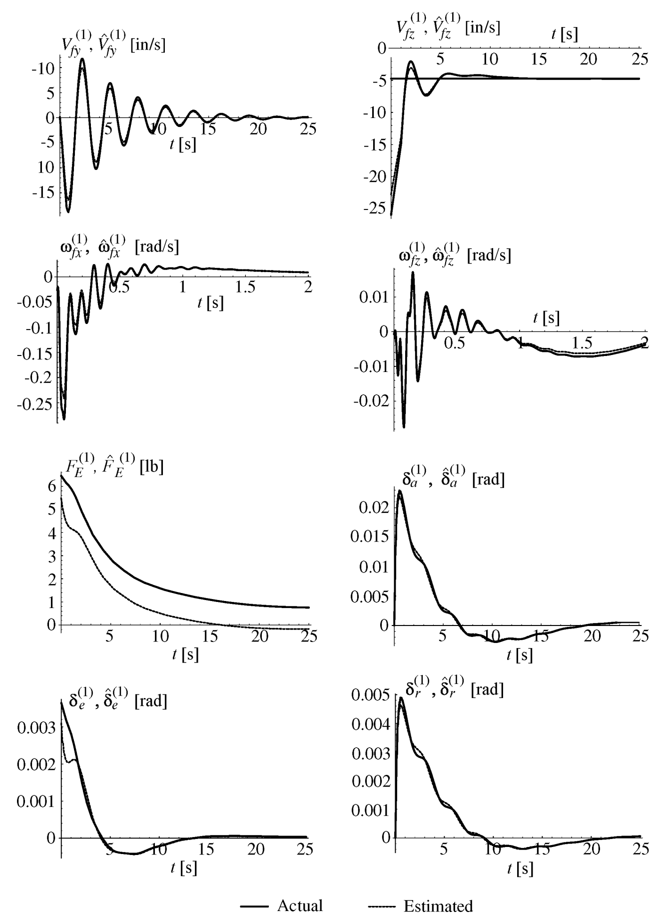

Fig. 4 Rigid-body velocities and control inputs.

B. Level Steady Turn Maneuver

We consider the case in which, in the zero-order problem, the aircraft flies at a constant velocity around a circular path of radius R in the horizontal X, Y plane. In this case, it is convenient to refer the rigid-body motions to a set of axes $x_1 y_1 z_1$ obtained through a rotation $\psi^{(0)}$ about Z, where $\dot{\psi}^{(0)} = \Omega = \text{constant}$. It is not difficult to see that axes x_1, y_1, and z_1 represent a set of cylindrical axes t, n, and Z, where t is tangent to the circle and n is normal to it. Denoting by $\dot{\bar{\boldsymbol{R}}}_f^{(0)}$ the velocity of O_f in terms of cylindrical components, the kinematical relation corresponding to the first of Eqs. (25) can be written as

$$\dot{\bar{\boldsymbol{R}}}_f^{(0)} = \begin{bmatrix} \dot{R}_{ft}^{(0)} & \dot{R}_{fn}^{(0)} & \dot{Z}^{(0)} \end{bmatrix}^T = [R\Omega \quad 0 \quad 0]^T = \bar{C}_f^{(0)T} \boldsymbol{V}_f^{(0)} \qquad (51)$$

where $\bar{C}_f^{(0)}$ is the matrix of direction cosines between tnZ and the fuselage body axes $x_f y_f z_f$; it is given in Ref. 24. Similarly, the second of Eqs. (25) can be written as

$$\dot{\boldsymbol{\theta}}_f^{(0)} = \begin{bmatrix} \dot{\phi}^{(0)} & \dot{\theta}^{(0)} & \dot{\psi}^{(0)} \end{bmatrix} = [0 \quad 0 \quad \Omega]^T = \left(E_f^{(0)}\right)^{-1} \boldsymbol{\omega}_f^{(0)} \qquad (52)$$

where $E_f^{(0)}$ is also given in Ref. 24. Equations (51) and (52) yield

$$\boldsymbol{V}_f^{(0)} = \begin{bmatrix} V_{fx}^{(0)} & V_{fy}^{(0)} & V_{fz}^{(0)} \end{bmatrix}^T = \bar{C}_f^{(0)} \dot{\bar{\boldsymbol{R}}}_f^{(0)}$$

$$= R\Omega \begin{bmatrix} c\theta^{(0)} & s\theta^{(0)} s\phi^{(0)} & s\theta^{(0)} c\phi^{(0)} \end{bmatrix}^T = \text{constant}$$

$$\boldsymbol{\omega}_f^{(0)} = \begin{bmatrix} \omega_{fx}^{(0)} & \omega_{fy}^{(0)} & \omega_{fz}^{(0)} \end{bmatrix}^T = E_f^{(0)} \dot{\boldsymbol{\theta}}_f^{(0)}$$

$$= \Omega \begin{bmatrix} -s\theta^{(0)} & c\theta^{(0)} s\phi^{(0)} & c\theta^{(0)} s\phi^{(0)} \end{bmatrix}^T = \text{constant} \qquad (53)$$

so that, from Eqs. (27), we have

$$\boldsymbol{p}_{Vf}^{(0)} = m\boldsymbol{V}_f^{(0)} + \tilde{S}^{(0)T} \boldsymbol{\omega}_f^{(0)} = \text{constant}$$

$$\boldsymbol{p}_{\omega f}^{(0)} = \tilde{S}^{(0)} \boldsymbol{V}_f^{(0)} + J^{(0)} \boldsymbol{\omega}_f^{(0)} = \text{constant} \qquad (54)$$

It follows that the equations of motion, the last two of Eqs. (25), reduce to

$$-\tilde{\omega}_f^{(0)} \boldsymbol{p}_{Vf}^{(0)} + \boldsymbol{F}^{(0)} = \boldsymbol{0}, \qquad -\tilde{V}_f^{(0)} \boldsymbol{p}_{Vf}^{(0)} - \tilde{\omega}_f^{(0)} \boldsymbol{p}_{\omega f}^{(0)} + \boldsymbol{M}^{(0)} = \boldsymbol{0} \qquad (55)$$

which are independent of time.

To determine the parameters defining the steady level turn maneuver, we choose the turn radius $R = 1.5\ \text{mile} = 7920$ ft and angular velocity $\Omega = 0.0526$ rad/s, so that $R\Omega = 416.67$ ft/s, and solve Eqs. (55) for the bank angle $\phi^{(0)}$, pitch angle $\theta^{(0)}$, and control vector $\boldsymbol{u}^{(0)} = [F_e^{(0)} \ \ \delta_a^{(0)} \ \ \delta_e^{(0)} \ \ \delta_r^{(0)}]^T$. Then, using Eqs. (53), we have

$$\boldsymbol{V}_f^{(0)} = 416.67 \begin{bmatrix} c\theta^{(0)} \\ s\theta^{(0)} s\phi^{(0)} \\ s\theta^{(0)} c\phi^{(0)} \end{bmatrix}, \qquad \boldsymbol{\omega}_f^{(0)} = 0.0526 \begin{bmatrix} -s\theta^{(0)} \\ c\theta^{(0)} s\phi^{(0)} \\ c\theta^{(0)} c\phi^{(0)} \end{bmatrix} \qquad (56)$$

We consider a flight at a 25,000-ft altitude, so that the speed of sound is 1016.1 ft/s and, hence, the Mach number is $416.67/1016.1 = 0.41$. Inserting Eqs. (56) into Eqs. (55) in conjunction with Eqs. (54) and solving the resulting transcendental equations, we obtain

$$\theta^{(0)} = 5.6494\ \text{deg}, \qquad \phi^{(0)} = 35.2942\ \text{deg}$$

$$F_E^{(0)} = 468.7429\ \text{lb}, \qquad \delta_a^{(0)} = -0.1604\ \text{deg}$$

$$\delta_e^{(0)} = -18.5237\ \text{deg}, \qquad \delta_r^{(0)} = -19.7384\ \text{deg} \qquad (57)$$

As in the case of steady level flight, the aircraft experiences static deformations in the steady level turn maneuver as well.

Next, we consider the first-order problem, Eqs. (28), with all quantities measured from the constant static solution, and once again we design controls by the LQR and LQG methods, in sequence. To

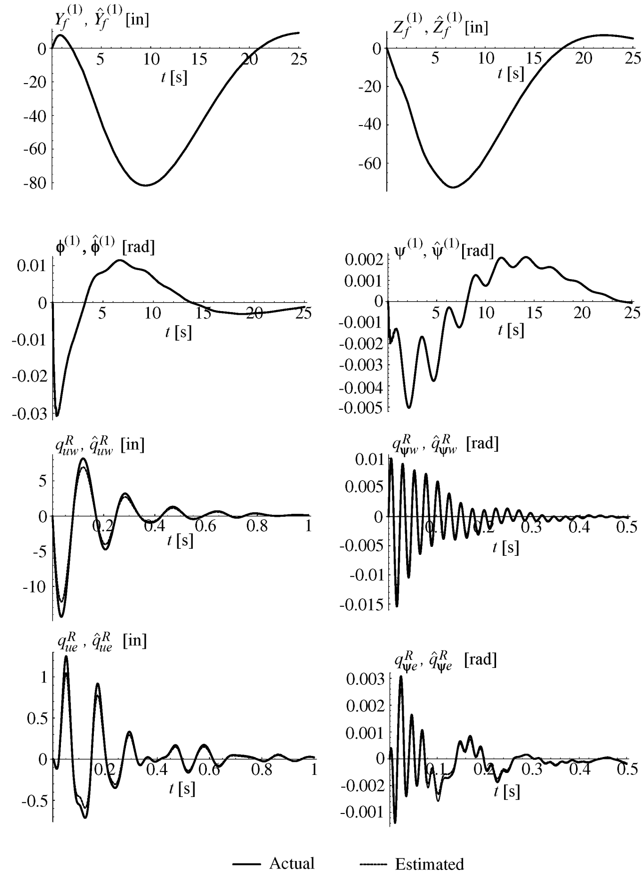

Fig. 5 Rigid-body and generalized displacements.

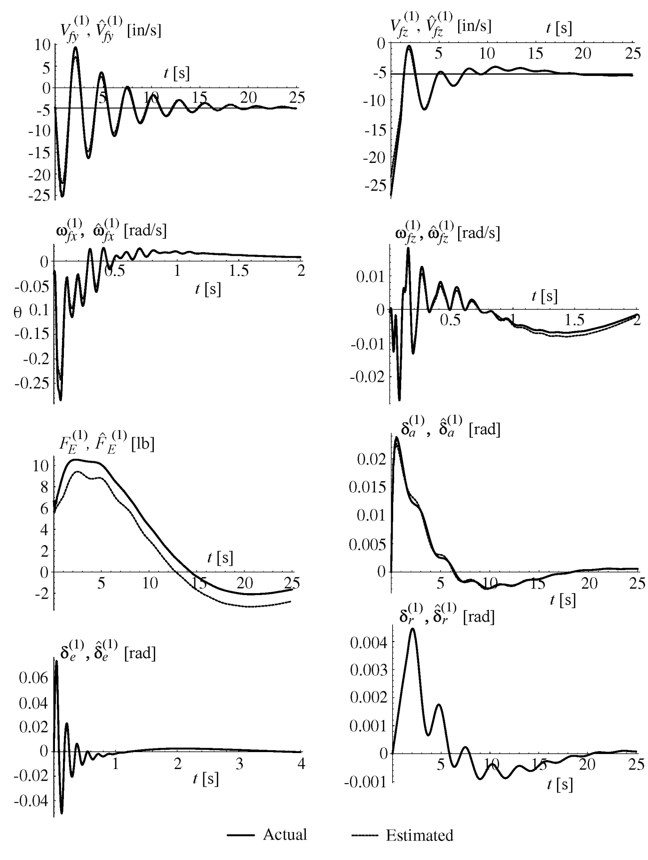

Fig. 6 Rigid-body velocities and control inputs.

this end, we use the state equation, Eq. (32), in which the feedback control vector has the form $\boldsymbol{u}^{(1)} = [F_E^{(1)} \quad \delta_a^{(1)} \quad \delta_e^{(1)} \quad \delta_r^{(1)}]^T$. In the case in which $\boldsymbol{u}^{(1)} = \boldsymbol{0}$ and $\boldsymbol{F}_{\text{ext}} = \boldsymbol{0}$, the state equations admit an exponential solution, yielding an eigenvalue problem. Solving the eigenvalue problem, we conclude that the system is unstable, with four eigenvalues being equal to zero and one being real and positive. Then, choosing appropriate weighting matrices for the performance measure, solving the corresponding steady-state Riccati equation, we obtain the gain matrix G. Solving the closed-loop eigenvalue problem, we obtain the closed-loop eigenvalues

$$\lambda_{1,2} = -0.1031 \pm 0.0870i, \qquad \lambda_3 = -0.2342$$
$$\lambda_{4,5} = -0.1194 \pm 0.2172i, \qquad \lambda_{6,7} = -0.2998 \pm 0.3089i,$$
$$\ldots, \lambda_{75,76} = -211.4850 \pm 1063.7465i \tag{58}$$

Clearly, all the closed-loop eigenvalues are either real and negative or complex with negative real part, so that the closed-loop first-order system is asymptotically stable. Hence, any disturbances from the steady level turn maneuver will be driven to zero.

Finally, we compute the response of the closed-loop system to the gust given by Eqs. (50). Figures 5 and 6 show a selected number of rigid-body and elastic variables, as well as the control inputs, both without and with an observer. Details of the observer design are given in Ref. 24. As can be seen, convergence of the estimated state and control variables to the actual ones is once again relatively fast.

VII. Conclusions

This work represents a new paradigm for the dynamics and control of maneuvering flexible aircraft. It reflects the perspective of a dynamicist using the system concept to produce a rigorous formulation of the problems associated with flexible flying machines. To this end, the theory integrates seamlessly into an online formulation all the necessary material from the areas of analytical dynamics, structural dynamics, aerodynamics, and controls. Based on fundamental principles, the formulation includes automatically all six rigid-body degrees of freedom and elastic deformations, as well as all the forces acting on the aircraft, namely, gravity, propulsion, aerodynamic and control forces, in addition to forces of an external nature, such as gusts. The seamless integration is achieved through the use of the same reference frame and the same variables to describe all the aircraft motions and forces acting on it. The unified formulation is made possible by equations of motion in terms of quasi-coordinates derived in earlier work by the first author of this paper. The formulation is modular in nature in the sense that the structural model, the aerodynamic theory, and the controls method can be replaced by any other ones to better suit different types of aircraft, provided certain criteria are satisfied, as discussed in a following paragraph. The only permanent part is the analytical dynamics, as manifested in the equations of motion in terms of quasi-coordinates. Even there, although the equations corresponding to the rigid-body variables remain the same, some modifications must be made in the equations corresponding to the elastic variables to reflect the type of elastic components involved.

At least in principle, the equations of motion, Eqs. (14), can be integrated numerically to determine the system response. In the case at hand, however, significant computational efficiency can be achieved by using some guidance from the way aircraft are flown. In particular, the rigid-body variables defining the flight path of the aircraft tend to be large, and the elastic deformations and the deviations from the flight path tend to be small. In view of this, it is possible to use a perturbation approach to separate Eqs. (14) into a flight dynamics problem for the rigid-body translations and rotations of the maneuvering aircraft and an extended perturbation problem for the elastic deformations and perturbations in the rigid-body variables, where the second problem is subject to inputs from the aircraft maneuvers. The flight dynamics problem is nonlinear and can be solved by inverse dynamics. On the other hand, the extended perturbation equations are linear but can be of high order, depending on the number of elastic degrees of freedom. The system of equations can be time-invariant or time-varying, depending on whether the inputs from the aircraft maneuvers are constant or time dependent. The extended perturbation equations can be used for feedback control design and, subsequently, for time simulations of the aircraft response. In the time-invariant case, they can also be used for a stability analysis. Note that the common aeroservoelasticity represents a special case of the extended perturbation problem. A numerical example presents a variety of time simulations of rigid-body perturbations and elastic deformations about 1) steady level flight and 2) a level steady turn maneuver.

The integration of the aerodynamics into the unified process is particularly challenging, and requires elaboration. For seamless integration, the aerodynamic forces must be referred to the same, generally noninertial, reference frame as that used for all other forces, and they must be expressed in terms of variables compatible with the variables used throughout the entire formulation. Moreover, because the simulation of the system response on a computer is carried out in discrete time, the size of the time step must be quite small, typically a small fraction of a second. The implication is that, to be ready to compute the state at the next sampling time, it is necessary to compute the aerodynamic forces within the time step. These two requirements are quite difficult to satisfy because most aerodynamic theories have been developed for purposes other than whole aircraft time response simulations, and the computation of the aerodynamic forces are notorious for consuming a great deal of time. In view of this, the development of a new aerodynamic method seems highly desirable. Such a method need not be unduly accurate because robust feedback controls are able to tolerate small errors in the aerodynamic forces. An aerodynamic theory satisfying the two requirements outlined earlier is strip theory. Even though strip theory may not be entirely satisfactory for describing the aerodynamic forces acting on whole aircraft, the method is often used in aircraft design. In fact, it is being used by the same company that provided the data for the Numerical Example section. The development of an aerodynamic technique suitable for online, or real-time response simulations is likely to require a great deal of time and effort. Although the use of a more suitable aerodynamic theory is desirable, it is not really necessary at this time. Indeed, at this time it is more important to demonstrate how the unified formulation works and how an eventual aerodynamic theory is to be integrated into the overall process. Certainly, such a demonstration can provide valuable guidance in the development of an appropriate aerodynamic method to be used in conjunction with whole aircraft time response simulations, thus helping reduce the time and effort required for the development of such a method.

Finally, it should be pointed out that, with appropriate modifications, the present formulation is eminently suited for UAVs and, in particular, for autonomous UAVs. It was applied here to an executive jet due to the ready availability of data from an actual flying aircraft. It should also be pointed out that all of the time response simulations presented here were carried out on a 1-GHz personal computer. This is particularly important for autonomous UAVs, which must be controlled by autopilots, because the required onboard computer is likely to be much closer to a personal computers than to a multiprocessor supercomputer.

The unified theory is expected to stimulate renewed interest in aeronautics research, ultimately providing a useful tool for the analysis and design of flexible aircraft.

Acknowledgments

This research was supported by the NASA Cooperative Agreement NCC-1-346 monitored by Martin R. Waszak, Dynamics and Control Branch, NASA Langley Research Center.

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